Communauté
UNIVERSITÉ Grenoble Alpes

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE GRENOBLE

Spécialité : **Mathématiques**

Arrêté ministériel : Arrêté du 7 août 2006 relatif à la formation doctorale

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préparée au sein **LAMA, Université Savoie Mont Blanc**
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Géométrie Tropicale et Systèmes Polynomiaux

Thèse soutenue publiquement le **22 Septembre 2016**,
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Acknowledgements

During this thesis, I have advanced a lot on both academic and personal levels. This three-year project came with the help of many people that participated directly or indirectly in my research.

Firstly, I would like to sincerely thank Frédéric Bihan for his huge help, and support in my PhD. Having immense patience to answer all my questions and to teach me different tools, he gave me a big push forward in my research. His guidance and our fruitful discussions gave me an important insight into my subject and in mathematics in general. I appreciate his massive efforts in correcting all my mistakes and helping me better write my papers and thesis. I feel very lucky having Frédéric as my advisor and mentor for his great personality inspired me even more to continue my journey in research.

I would like to thank Erwan Brugallé and J. Maurice Rojas for accepting to be examiners of my thesis, as well as Georges Comte, Ilia Itenberg, Krzysztof Kurdyka and Thorsten Theobald for accepting to be a part of my jury. I also appreciate the very helpful examples of particular polynomial systems which were provided by J. Maurice Rojas, and the much valuable computations made by Pierre-Jean Spaenlehauer both of which confirmed that my results are on the right track.

I owe my gratitude also to Michel Raibaut who always re-encouraged me during the common tough moments in research. The discussions we had answering my questions gave me a broader perspective in mathematics, and our hikes, dinners and all the activities we did have made my stay in France even much more enjoyable. I am very happy to have met such a great friend and colleague.

My old friends Emile, Hassan, Hisham, Ralph and the people that I became friends with during my stay in France made a firm positive impact on my thesis and my life. Each in their own way, I appreciate the great friendships of my colleagues Charlotte and Rodrigo which gave me a big support with all the fun and fascinating time spent together. Thanks to Pierre for interesting discussions and Rodolphe for helpful computations. The enjoyable companies of my peers Beni, Bilal, Clovis, Joseph, Lama, Marion and Rémy in the offices of PhD students made a friendly and satisfying ambience to work in.

I would like to thank all the members of our geometry group for the friendly ambience. I am grateful to Olivier Le Gale for interesting discussions and support. I would like to acknowledge the help of Christian Bourdarias, Philippe Briand, Georges Comte and Stéphane Simon for their various forms of help during my thesis. I appreciate also the beautiful environment of whole math department that is provided by all the nice people that are parts of it.

My biggest motivation and support comes from my parents and my little brother Albertik all of whom always encourage and are proud of me.

Last but not the least, a very special thanks to a very special person, Rona who made my life easier and my days brighter…

Résumé

Géométrie Tropicale et Systèmes Polynomiaux

Les systèmes polynomiaux réels sont omniprésents dans de nombreux domaines des mathématiques pures et appliquées. A. Khovanskii a fourni une borne *fewnomiale* supérieure sur le nombre de solutions positives non-dégénérées d'un système polynomial réel de n équations à n variables qui ne dépend que du nombre de monômes apparaissant dans les équations. Cette dernière borne a été récemment améliorée par F. Bihan et F. Sottile, mais la borne résultante peut être encore améliorée, même dans certains cas simples.

Le but de ce travail est d'aborder trois problèmes importants dans la théorie des Fewnomials. Considérons une famille de systèmes polynomiaux réels avec une structure donnée (par exemple, support ou le nombre de monômes). Un problème est de trouver de bonnes bornes supérieures pour leurs nombres de solutions réelles (ou positives). Un autre problème est de construire des systèmes dont le nombre de solutions réelles (ou positives) sont proches de la meilleure borne supérieure connue. Lorsqu'une borne supérieure optimale est bien connue, qu'est ce qu'on peut dire dans le cas où elle est atteinte?

Dans cette thèse, nous affinons un résultat de M. Avendaño en démontrant que le nombre de points d'intersection réels d'une droite réelle avec une courbe réelle plane définie par un polynôme avec au plus t monômes est soit infini ou ne dépasse pas $6t-7$. En outre, on montre que notre borne est optimale pour $t=3$ en utilisant les dessins d'enfant réels de Grothendieck. Cela montre que le nombre maximal de points d'intersection réels d'une droite réelle avec une courbe trinomiale réelle plane est onze.

Nous considérons ensuite le problème de l'estimation du nombre maximal de points d'intersection transverses positifs d'une courbe plane trinomiale et d'une courbe plane t-nomiale. T-Y Li, J.-M. Rojas et X. Wang ont montré que ce nombre est borné par 2^t-2, et récemment P. Koiran, N. Portier et S. Tavenas ont trouvé la borne supérieure $2t^3/3+5t$. Nous fournissons la borne supérieure $3\cdot 2^{t-2}-1$ qui est optimale pour $t=3$ et est la plus petite pour $t=4,\ldots,9$. Ceci est réalisé en utilisant la notion de dessins d'enfant réels. De plus, nous étudions en détail le cas $t=3$ et nous donnons une restriction sur les supports des systèmes atteignant la borne optimale cinq.

Un circuit est un ensemble de $n+2$ points dans $\mathbb{R}^n$ qui sont minimalement affinement dépendants. Il est connu qu'un système supporté sur un circuit a au plus $n+1$ solutions positives non dégénérées, et que cette borne est optimale. Nous utilisons les dessins d'enfant réels et le *patchwork combinatoire* de Viro pour donner une caractérisation complète des circuits supportant des systèmes polynomiaux avec le nombre maximal de solutions positives non dégénérées.

Nous considérons des systèmes polynomiaux de deux équations à deux variables avec cinq monômes distincts au total. Ceci est l'un des cas les plus simples où la borne supérieure optimale sur le nombre de solutions positives non dégénérées n'est pas connue. F. Bihan et F. Sottile ont prouvé que cette borne optimale est majorée par quinze. D'autre part, les meilleurs exemples avaient seulement cinq solutions positives non dégénérées.

Nous considérons des systèmes polynomiaux comme avant, mais défini sur le corps des *séries de Puiseux réelles généralisées et localement convergentes*. Les images par l'application de valuation des solutions d'un tel système sont des points d'intersection de deux courbes tropicales planes. En utilisant des intersections non transverses des courbes tropicales planes, on obtient une construction d'un système polynomial réel comme ci-dessus ayant sept solutions positives non dégénérées.

Mots clés— Géométrie Algébrique Réelle, Théorie des Fewnomials, Géométrie Tropicale, Systèmes Polynomiaux

Abstract

Tropical Geometry and Polynomial Systems

Real polynomial systems are ubiquitous in many areas of pure and applied mathematics. A. Khovanskii provided a *fewnomial* upper bound on the number of non-degenerate positive solutions of a real polynomial system of n equations in n variables that depends only on the number of monomials appearing in the equations. The latter bound was recently improved by F. Bihan and F. Sottile, but the resulting bound still has room for improvement, even in some simple cases.

The aim of this work is to tackle three main problems in Fewnomial theory. Consider a family of real polynomial systems with a given structure (for instance, supports or number of monomials). One problem is to find good upper bounds for their numbers of real (or positive) solutions. Another problem is to construct systems whose numbers of real (or positive) solutions are close to the best known upper bound. When a sharp upper bound is known, what can be said about reaching it?

In this thesis, we refine a result by M. Avendaño by proving that the number of real intersection points of a real line with a real plane curve defined by a polynomial with at most t monomials is either infinite or does not exceed $6t - 7$. Furthermore, we prove that our bound is sharp for $t = 3$ using Grothendieck's *real dessins d'enfant*. This shows that the maximal number of real intersection points of a real line with a real plane trinomial curve is eleven.

We then consider the problem of estimating the maximal number of transversal positive intersection points of a trinomial plane curve and a t-nomial plane curve. T-Y Li, J.-M. Rojas and X. Wang showed that this number is bounded by $2^t - 2$, and recently P. Koiran, N. Portier and S. Tavenas proved the upper bound $2t^3/3 + 5t$. We provide the upper bound $3 \cdot 2^{t-2} - 1$ that is sharp for $t = 3$ and is the tightest for $t = 4, \dots, 9$. This is achieved using the notion of real dessins d'enfant. Moreover, we study closely the case $t = 3$ and give a restriction on the supports of systems reaching the sharp bound five.

A *circuit* is a set of $n + 2$ points in $\mathbb{R}^n$ that is minimally affinely dependent. It is known that a system supported on a circuit has at most $n + 1$ non-degenerate positive solutions, and that this bound is sharp. We use real dessins d'enfant and Viro's *combinatorial patchworking* to give a full characterization of circuits supporting polynomial systems with the maximal number of non-degenerate positive solutions.

We consider polynomial systems of two equations in two variables with a total of five distinct monomials. This is one of the simplest cases where the sharp upper bound on the number of non-degenerate positive solutions is not known. F. Bihan and F. Sottile proved that this sharp bound is not greater than fifteen. On the other hand, the best examples had only five non-degenerate positive solutions. We consider polynomial systems as before, but defined over the field of *real generalized locally convergent Puiseux series*. The images by the valuation map of the solutions of such a system are intersection points of two plane tropical curves. Using non-transversal intersections of plane tropical curves, we obtain a construction of a real polynomial system as above having seven non-degenerate positive solutions.

Keywords— Real Algebraic Geometry, Theory of Fewnomials, Tropical Geometry, Polynomial Systems

Contents

Chapter 1

Introduction

One of the fundamental problems in mathematics is solving real polynomial equations since polynomial systems arise naturally and ubiquitously in mathematics and many of its applications. We see them appearing in such fields as control theory [Byr89], kinematics [BR90], chemistry [GH02, MFR$^+$16] and many others where it is mainly the real solutions that matter. In this introduction we give a brief overview on solving polynomial equations and state the main results of this thesis. For a more detailed exposition on solving polynomial equations, see for example [Sot11] or [Stu02].

1.1 Univariate polynomials

Galois theory shows that for a univariate polynomial f with real coefficients and degree less or equal to four, there exists a general formula that explicitly determines the complex roots of f in terms of its coefficients. However this statement is false if f has degree larger than four. This means that computing the roots of high-degree polynomials is not an easy task. Nevertheless, there are many methods and results devoted especially to this problem (see for example [Stu02]). By the *Fundamental theorem of algebra*, any univariate polynomial f has at least one complex root. Moreover, the number of its complex roots (counted with multiplicities) is equal to its degree.

Unfortunately, in general the degree is a bad estimate for the number of real roots of f e.g. $1 - x^{100}$ has 98 non-real roots and only two real ones. Descartes' rule of sign [Des97], which dates back to 1637, is one of the earliest results that gives a more accurate estimation for the number of real roots of f. Suppose that we write the terms of f in increasing order of their exponents,

$$f(x) = b_0x^{k_0} + b_1x^{k_1} + \cdots + b_mx^{k_m}, \tag{1.1.1}$$

where $b_i \neq 0$ and $k_0 < \cdots < k_m$.

Theorem 1.1 (Descartes' rule of sign)**.** *The number r of isolated positive roots of f, counted with multiplicity, is at most the number of sign changes of its coefficients,*

$$r \leq \{i \mid 1 \leq i \leq m \quad \textit{and} \quad b_{i-1}b_i < 0\}.$$

Theorem 1.1 also holds true for univariate polynomials with real exponents. The immediate consequence for this rule is that the number of positive solutions of f is bounded from above by

m. Moreover, replacing x by $-x$ and applying Theorem 1.1 to the resulting polynomial gives a similar estimation for the number of negative roots of f. Therefore, the number of non-zero real roots of f is less or equal to $2m$.

It is important to note that Descartes' rule of sign, and thus the resulting Descartes' bound, is independent of the degree. This naturally brings about the question of generalizing Theorem 1.1 to a polynomial system.

1.2 Sparse polynomial systems

Consider a real polynomial system

$$f_1(z_1, \ldots, z_n) = \cdots = f_n(z_1, \ldots, z_n) = 0. \tag{1.2.1}$$

In general, we look for solutions of (1.2.1) in the complex torus $(\mathbb{C}^*)^n$ since solutions in coordinate hyperplanes are solutions in complex tori of smaller dimensions of truncated systems. A solution $\zeta \in \mathbb{C}^n$ of (1.2.1) is **non-degenerate** if the Jacobian of (1.2.1) evaluated at ζ has full rank. Non-degenerate solutions are easier to manipulate since their number will not decrease after any "slight" perturbation of the coefficients of the associated system.

1.2.1 Polyhedral bounds

Denote by d_i the total degree of f_i. Bézout's fundamental Theorem [Béz79] states that the number of non-degenerate complex solutions of (1.2.2) is less or equal to $d_1 \cdots d_n$. Moreover, this bound is sharp. Polynomial systems that arise naturally may have some special structure, for instance in terms of disposition of the exponent vectors or their number (cf. [Sot11]). However, a great part of this combinatorial data is disregarded when using the degree to bound the number of complex solutions, and thus the Bézout bound can be rough. In fact, there exist bounds that depend on the polyhedral structure associated to the polynomial system that we describe now.

To any $w = (w^1, \ldots, w^n) \in \mathbb{Z}^n$ is associated a monomial $z^w \in \mathbb{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Consider a Laurent polynomial $f \in \mathbb{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ written as

$$f(z) := \sum_{w \in \mathcal{W}} c_w z^w, \tag{1.2.2}$$

where $c_w \neq 0$ for all $w \in \mathcal{W}$. The set $\mathcal{W}$ is called the **support** of f. The support of a system (1.2.1) is the union of the supports of $f_1, \ldots, f_n$. The **Newton polytope** of f is the convex hull $\Delta_\mathcal{W}$ of $\mathcal{W}$. Write $\mathrm{Vol}(\Delta)$ for the Euclidean volume of a polytope $\Delta \subset \mathbb{R}^n$. We have the following fundamental result due to A. Kushnirenko [Kus75].

Theorem 1.2 (Kushnirenko)**.** *If* (1.2.1) *has support* $\mathcal{W}$*, then it has at most* $n!\,\mathrm{Vol}(\Delta_\mathcal{W})$ *isolated solutions in* $(\mathbb{C}^*)^n$*, and exactly this number if the polynomials are generic among systems with support* $\mathcal{W}$*.*

D. N. Bernstein [Ber75] refined this result taking the individual supports into account. Let $\mathcal{W}_i$ denotes the support of the polynomial f_i appearing in (1.2.1). The **Minkowski sum** of the convex hulls of $\mathcal{W}_i$ for $i = 1, \ldots, n$, is a pointwise sum

$$\Delta_{\mathcal{W}_1} + \cdots + \Delta_{\mathcal{W}_n} = \{w_1 + \cdots + w_n \mid w_1 \in \Delta_{\mathcal{W}_1}, \ldots, w_n \in \Delta_{\mathcal{W}_n}\}.$$

Minkowski (see [Ewa12]) showed that given convex bodies $K_1, \dots, K_n$ in $\mathbb{R}^n$ and positive numbers $\lambda_1, \dots, \lambda_n$, the function $\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ is a homogeneous polynomial in $\lambda_1, \dots, \lambda_n$ of degree n, so there exist coefficients $V(K_{i_1}, \dots, K_{i_n})$ for $i_1, \dots, i_n \in [n]$ such that

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \dots, i_n \in [n]} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{1.2.3}$$

The **mixed volume**, $\mathrm{MV}(K_1, \dots, K_n)$ of $K_1, \dots, K_n$ is $V(K_1, \dots, K_n)$. Now we state Bernstein's important generalization of Kushnirenko's Theorem.

Theorem 1.3 (Bernstein). *A system of n polynomials in n variables where the polynomials have support $\mathcal{W}_1, \dots, \mathcal{W}_n$ has at most $MV(\Delta_{\mathcal{W}_1}, \dots, \Delta_{\mathcal{W}_n})$ isolated solutions in $(\mathbb{C}^*)^n$, and exactly this number when the polynomials are generic for their given supports.*

It is worth noting that a non-degenerate solution of a system is an isolated one, thus both Kuschnirenko and Bernstein Theorems give upper bounds for the number of non-degenerate solutions in $(\mathbb{C}^*)^n$ of a polynomial system. Although the degree and previous polyhedral bounds hold true for the number of non-degenerate solutions in $(\mathbb{R}^*)^n$ as well, the resulting bounds are not always sharp. This typically happens when the total support $\mathcal{W}$ of (1.2.1) has few elements comparatively to $\Delta_{\mathcal{W}} \cap \mathbb{Z}^n$.

1.2.2 Fewnomial bounds

Denote by $\mathcal{W} \subset \mathbb{R}^n$ the support of (1.2.1). Multivariate generalizations of Descartes' bound (Theorem 1.1) for systems of multivariate polynomials are called **Fewnomial bounds**[1]. A particular attention is paid to the positive solutions of (1.2.1), which are the solutions contained in the positive orthant of $\mathbb{R}^n$. Indeed, assume that there exists a sharp upper bound $N_{\mathcal{W}}$ on the number of non-degenerate positive solutions of (1.2.1) that depends only on $\mathcal{W}$. Then this $N_{\mathcal{W}}$ also bounds the number of solutions contained in any other orthant, and thus (1.2.1) will not have more than $2^n N_{\mathcal{W}}$ solutions in $(\mathbb{R}^*)^n$. Recall that Descartes showed that we have $N_{\mathcal{W}} = |\mathcal{W}| - 1$ for $n = 1$, but still, before Khovanskii's book [Kho91], it was not clear that such $N_{\mathcal{W}}$ even exists for any $n \geq 2$.

Theorem 1.4 (Khovanskii). *A system of n real polynomials in n variables involving $n + k + 1$ distinct monomials has fewer than*

$$2^{\binom{n+k}{2}} (n+1)^{n+k}. \tag{1.2.4}$$

non-degenerate positive solutions.

The existence of a bound on the number of non-degenerate positive solutions that is independent of the degrees of the polynomials was revolutionary and is the main point of Khovanskii's result. It also confirms Kushnirenko's principle that the topological complexity of objects, defined by real-valued polynomials, can be controlled by the complexity of the definition of these polynomials rather than by degrees or by some characteristics of Newton polyhedra of equations.

Also, the bound in Theorem 1.4 is not sharp. In fact, Theorem 1.4 is a particular case of a Khovanskii's more general result involving solutions in $\mathbb{R}^n$ of polynomial functions in logarithms of the coordinates and monomials (see [Kho91]). For example, when $k = 0$, the support $\mathcal{W}$ of the system is a simplex, and there will be at most *one* real solution, which is smaller than $2^{\binom{n}{2}}(n+1)^n$.

[1]The term "Fewnomial" was coined by A. Kushnirenko, where he replaced the term "poly" of the word "polynomial", by the term "Few" (c.f. [Kus08])

Although it was commonly believed that Khovanskii's bound (1.2.4) was far from being sharp, improving it turns out to be not an easy task.

Fewnomial theory was mainly initiated by Kushnirenko's famous conjecture which was formulated in the late 70's as a tentative generalization of Descartes' bound.

Conjecture 1.5 (Kushnirenko). *A system of n real polynomials in n variables, where the polynomials have supports $\mathcal{W}_1, \ldots, \mathcal{W}_n$, has at most*

$$\prod_{i=1}^{n}(|\mathcal{W}_i| - 1)$$

non-degenerate positive solutions.

Constructing polynomial systems reaching Kushnirenko's conjectured bound is not a difficult task. Namely, such a construction might be for instance a system

$$g_i(z_i) = 0, \quad \text{for} \quad i = 1, \ldots, n$$

consisting of univariate polynomials, where each g_i has m_i terms and $m_i - 1$ non-degenerate positive solutions (Descartes' bound). In fact, the lack of efficient construction methods at the time instigated Kushnirenko to establish his conjecture.

1.3 Results prior to this thesis

After the famous Khovanskii's Theorem, there were many recent contributions dedicated to the theory of Fewnomials, (c.f. [Sot11] for a survey). In this section, we give but a few of the many results developed in this millennia. Most of these results are further investigated and in some cases improved in this thesis.

1.3.1 Around Khovanskii's bound

Consider a real polynomial system

$$f_1(z) = \cdots = f_n(z) = 0 \tag{1.3.1}$$

in n variables supported on a set $\mathcal{W} \subset \mathbb{Z}^n$ such that $|\mathcal{W}| = n + k + 1$ for some $k \geq 1$. In [BS07], F. Bihan and F. Sottile significantly reduced Khovanskii's fewnomial bound (1.2.4) by showing that there are fewer than

$$\frac{e^2 + 3}{4} 2^{\binom{k}{2}} n^k \tag{1.3.2}$$

non-degenerate positive solutions to (1.3.1). The method they used consists of reducing the original system to a system of k equations in k variables, called *Gale transform*. This Gale transform depends upon the vector configuration "Gale" dual to the exponents of the monomials in the original system (see [BS08]). This reduction gives that an upper bound on the Gale transform also holds true for the number of solutions of (1.3.1). The bound in (1.3.2) also holds true for polynomials with real exponents. Moreover, the significance of it is that (1.3.2) is asymptotically sharp in the sense that for fixed k, there are systems with $O(n^k)$ positive solutions [BRS08] .

The constant $\frac{e^2+3}{4}$ appearing in (1.3.2) is artificial, its purpose is only to bound from above a more complicated expression. Moreover, the authors in [BS07] believe that the term $2^{\binom{k}{2}}$ in (1.3.2) is considerably overstated. In fact, when $k = 2$, this smaller bound (1.3.2) is actually $2n^2 + \lfloor \frac{(n+3)(n+1)}{2} \rfloor$, and when $n = 2$ it is 15. Note that when plugging $n = k = 2$ in (1.2.4), we obtain $2^6 \cdot 3^4 = 5184$. Although the new bound 15 is a considerably smaller fewnomial bound for a system where $n = k = 2$, the authors of [BS07] maintain that the sharp bound is still smaller. The case $n = k = 2$ is the first case where we do not know much about. In fact, prior to this thesis, the first known construction, giving a lot of non-degenerate positive solutions of a system of two polynomials in two variables with five monomials was essentially that of B. Haas (1.3.5). Such a construction gives five non-degenerate positive solutions, and shows that the sharp upper bound on the number of non-degenerate positive solutions is greater or equal to 5. Later on, we will call a system of two equations in two variables with 5 distinct monomials a system of type $n = k = 2$.

1.3.2 Using combinatorial patchworking

Consider a system

$$f_{1,t}(z) = \cdots = f_{n,t}(z) = 0, \tag{1.3.3}$$

where each polynomial of (1.3.3) is obtained from a polynomial $\sum_w c_w z^w$ of (1.3.1) by multiplying each monomial $c_w z^w$ by some real power of t, where t is a positive parameter that will be taken close to zero. Let $V(f_{i,t})$ denote the zero set of $f_{i,t}$ in $\mathbb{R}^n$. For any $\epsilon \in \{\pm 1\}^n$, consider the orthant

$$(\mathbb{R}_{>0})^\epsilon := \{x \in \mathbb{R}^n \mid x_i \epsilon_i > 0 \quad i = 1, \ldots, n\},$$

and let $V_\epsilon(f_{i,t})$ be the intersection of $V(f_{i,t})$ with $(\mathbb{R}_{>0})^\epsilon$.

O. Viro's Theorem states that one can construct combinatorially a space Q_ϵ together with a simplicial complex $\mathcal{Z}_\epsilon \subset Q_\epsilon$ such that the couple $(Q_\epsilon,\ \mathcal{Z}_\epsilon)$ is homeomorphic to $((\mathbb{R}_{>0})^\epsilon,\ V_\epsilon(f_{i,t}))$ for $t > 0$ small enough. From this, one can recover (up to homeomorphisms) the whole hypersurface $V(f_{i,t})$ (for $t > 0$ small enough) by gluing its different parts together with their ambient spaces.

This was generalized by B. Sturmfels [Stu94] for any complete intersection $V(f_{1,t}) \cap \cdots \cap V(f_{s,t})$, with $s \leq n$, given that the exponents of t are "sufficiently generic". When $s = n$, this method can be used to construct systems with many non-degenerate positive solutions and given supports. Recently, F. Bihan [Bih14] gave a bound on the number of non-degenerate real solutions that are constructed using Sturmfels' generalization of Viro's Theorem. This bound is given by the so-called *discrete mixed volume* of the supports of $f_{i,t}$. In fact, he proved that this bound is smaller than the one given in Kushnirenko's conjecture (see Subsection 1.3.4). When $n = 2$ and $k = 1$, the discrete mixed volume is not larger than 3 and the corresponding bound is sharp (see Subsection 1.3.3). When $n = k = 2$, it is easy to compute that the discrete mixed volume is not larger than 6 (see Lemma 6.4 in Chapter 6), and it is not known if the corresponding bound is sharp.

1.3.3 Systems supported on a circuit

One of the first non-trivial cases arises when $n \geq 2$ and $k = 1$, in which case the support $\mathcal{W}$ of (1.3.1) is a set of $n + 2$ points in $\mathbb{R}^n$. F. Bihan [Bih07] proved that any polynomial system supported on such $\mathcal{W}$ has at most $n + 1$ non-degenerate positive solutions and that this bound is sharp. Moreover, if this bound is reached, then $\mathcal{W}$ is minimally affinely dependent, which means that it is a *circuit* in $\mathbb{R}^n$. Polynomial systems supported on a circuit in $\mathbb{Z}^n$ whose all non-degenerate complex solutions are positive have been studied in [Bih15] (such systems are called *maximally positive*). As a main result, it is given for any positive integer n a finite list of circuits in $\mathbb{Z}^n$ that can support maximally positive systems up to the obvious action of the group of invertible integer affine transformations of $\mathbb{Z}^n$.

Also for the circuit case, F. Bihan and A. Dickenstein [BD16] presented the first multivariate version of Descartes' rule of signs to bound the number of positive real solutions of a system supported on a circuit, in terms of the sign variation of a sequence associated to both the exponent vectors and the given coefficients. In fact, it is also shown that the bound they gave is sharp and is related to the signature of the circuit.

The first time that Grothendieck's *real dessins d'enfant*, which are graphs embedded on the Riemann sphere, were used in the fewnomial context was due to F. Bihan [Bih07]. Namely, he uses dessins d'enfant to show the sharpness of the bound $n+1$ for the number of positive solutions of a system supported on a circuit $\mathcal{W} \subset \mathbb{R}^n$. He also proves using the same technique the sharpness of bounds for the number of *real* solutions of such systems. As it turns out, if one can reduce a fewnomial system to a rational polynomial function $\mathbb{C}P^1 \to \mathbb{C}P^1$, then one can hope to use real dessins d'enfant in a fruitful way to closely study the original system. This technique gives an interesting point of view on constructing polynomial systems with a large number of real solutions (see Chapter 3), characterizing such systems (see Chapter 5) and even bounding the number of positive solutions of sparse polynomial systems (see Chapter 4).

Sturmfels' version of Viro's *combinatorial patchworking* is yet another effective technique from real algebraic geometry that can be used to construct polynomial systems with many real solutions. This generalisation [Stu94] is for complete intersections of real algebraic hypersurfaces. Among many other implementations in fewnomials, it was used by K. Phillipson and J.-M. Rojas [PR13, proof of Lemma 1.8] to construct a polynomial system over local fields supported on a circuit that has $n+1$ positive solutions.

1.3.4 Around Kuschnirenko's conjecture

Consider the system (1.3.1), and for $i = 1, \ldots, n$, denote by m_i the number of points contained in the support of f_i. Recall that Kushnirenko' Conjecture 1.5 states that (1.3.1) cannot have more than

$$\prod_{i=1}^{n}(m_i - 1)$$

non-degenerate positive solutions.

1.3.4.1 First counterexamples

The conjectural bound is not a bound on the number of isolated positive solutions. W. Fulton gave a counterexample in [Ful13] that goes as follows (see also [Stu02]). Consider the system

$$\prod_{i=1}^{m}(z_1 - i)^2 + \prod_{i=1}^{m}(z_2 - i)^2 = 0, \quad z_1(z_3 - 1) = 0, \quad z_2(z_3 - 1) = 0, \tag{1.3.4}$$

where $m \geq 5$. Kushnirenko's Conjecture predicts that such a system has at most $(4m+1-1)(2-1)(2-1) = 4m$ real positive solutions. However there are m^2 positive solutions of (1.3.4) of the form $(i, j, 1)$, for $i, j \in \mathbb{N}^*$ between 1 and m.

A particular case of A. Kuchnirenko's conjecture states that when $n = 2$ and $m_1 = m_2 = 3$, the system (1.3.1) has at most four non-degenerate positive solutions. In an effort to disprove this conjecture, Haas had shown in [Haa02] that

$$10x^{106} + 11y^{53} - 11y = 10y^{106} + 11x^{53} - 11x = 0 \tag{1.3.5}$$

has five non-degenerate positive solutions. Konstantin A. Sevastyanov, a colleague of Kushnirenko, had found a similar counter-example much earlier. Unfortunately, this counterexample does not seem to have been recorded and, tragically, Sevastyanov died before publishing his counterexample.

It was later shown in [LRW03] using a case by case analysis that when $n = 2$ and $m_1 = m_2 = 3$, the sharp bound on the number of non-degenerate positive solutions is five. Moreover, it was proved in the same paper that if this bound is reached, then the Minkowski sum of the associated Newton polytopes Δ_1 and Δ_2 is an hexagon.

A simpler polynomial system

$$x^6 + (44/31)y^3 - y = y^6 + (44/31)x^3 - x = 0, \tag{1.3.6}$$

that also has five positive solutions was discovered by A. Dickenstein, J.-M. Rojas, K. Rusek and J. Shih [DRR07]. In addition, they showed that such systems are rare in the following sense. They study the discriminant variety of coefficients spaces of the polynomial system

$$x^{2d} + ay^d - y = y^{2d} + bx^d - x = 0, \tag{1.3.7}$$

with parameters (a, b, d), and show that the chambers (connected components of the complement) containing systems with the maximal number of positive solutions are small.

1.3.4.2 A trinomial and a t-nomial

Real polynomial systems in two variables

$$f = g = 0, \tag{1.3.8}$$

where f has $t \geq 3$ non-zero terms and g has three non-zero terms have been studied by T.Y. Li, J.-M. Rojas and X. Wang [LRW03]. They showed that such a system, allowing real exponents, has at most $2^t - 2$ isolated positive solutions. The idea is to substitute one variable of the trinomial in terms of the other, and thus one can reduce the system to an analytic function in one variable

$$h(x) = \sum_{i=1}^{t} a_i x^{k_i} (1 - x)^{l_i},$$

where all the coefficients and exponents are real. The number of positive solutions of (1.3.8) is equal to that of $h = 0$ contained in $]0, 1[$. The main techniques used in [LRW03] are an extension of Rolle's Theorem and a recursion involving derivatives of certain analytic functions. In fact, the results of Li, Rojas and Wang [LRW03] are more general. Consider a polynomial system

$$f_1 = \cdots = f_n = 0 \tag{1.3.9}$$

in n variables, where the functions $f_1, \ldots, f_{n-1}$ are trinomials and f_n has t distinct monomials. The authors in [LRW03] show that (1.3.9) has at most $n + n^2 + \cdots + n^{t-1}$ non-degenerate positive solutions.

The exponential upper bound $2^t - 2$ on the number of positive solutions of (1.3.8) has been recently refined by P. Koiran, N. Portier and S. Tavenas [KPT15b] into a polynomial one. They considered an analytic function in one variable

$$\sum_{i=1}^{t} \prod_{j=1}^{m} f_j^{\alpha_{i,j}}, \tag{1.3.10}$$

where all f_j are real polynomials of degree at most d and all the powers of f_j are real. Using the Wronskian of analytic functions, it was proved that the number of positive roots of (1.3.10) in an interval I (assuming that $f_j(I) \subset]0, +\infty[$) is equal to $\frac{t^3 md}{3} + 2tmd + t$. As a particular case (taking $m = 2$, $d = 1$ and $I =]0, 1[$), they obtain that $h(x) = \sum_{j=1}^{t} a_i x^{k_i} (1-x)^{l_i}$ has at most $2t^3/3 + 5t$ roots in I.

1.3.4.3 A plane curve and a line

Interestingly, when the trinomial g of (1.3.8) is a linear polynomial, then the sharp bound on the number of non-degenerate real solutions of (1.3.8) is a linear function in t.

Namely, M. Avendaño showed in [Ave09] that such a system has either an infinite number or at most $6t - 6$ solutions in $(\mathbb{R}^*)^2$, where the latter ones are counted with multiplicities. In particular, he proved that the number of non-degenerate *positive* solutions of the latter system is at most $2t - 2$. The method used in [Ave09] consists of substituting z_2 by $az_1 + b$ in (1.3.8) for some non-zero real numbers a and b. This way, with the help of Descartes' rule of sign applied to the resulting univariate polynomial, one eventually obtains the bound $2t - 2$.

1.3.5 Around a polynomial-fewnomial conjecture

A. Kushnirenko also formulated the following conjecture (see [Kus08] for more background). Consider a system

$$f(x, y) = g(x, y) = 0 \tag{1.3.11}$$

of two equations in two variables, where g is a polynomial with t distinct monomial terms, and f is a polynomial of degree d.

Conjecture 1.6. *The system* (1.3.11) *has at most $N(d, t)$ non-degenerate positive solutions, where $N(d, t)$ is a function depending only on the numbers d and t.*

Sevostyanov showed in 1978 that such $N(d, t)$ exists. However, his result (together with his counterexample to Kushnirenko's conjecture) was never published. According to [Sot11], this result was the inspiration for Khovanskii to develop his theory of fewnomials.

Clearly, by Khovanskii and Bihan-Sottile bounds, this $N(d, t)$ exists, however since (1.3.11) is a very particular case of the generic system (1.2.1), bounds (1.2.4) and (1.3.2) (which are exponential in d and t) might be too large. M. Avendaño's previously-discussed bound [Ave09] shows that $N(1, t) \leq 2t - 2$, which turns out to be a sharp bound for $t = 3$ (see [BEH15]).

The smallest bound so far for any values d and t was discovered by P. Koiran, N. Portier and S. Tavenas [KPT15a]. They showed that (1.3.11) has only $O(d^3 t + d^2 t^3)$ real solutions when it has a finite number of real solutions. Moreover, if the set of real solutions is infinite then it has at most $O(d^3 t + d^2 t^3)$ connected components.

1.4 Results of the thesis

We divide our main results into four chapters.

1.4.1 Chapter 3: Intersecting a sparse plane curve and a line

Chapter 3 is a joint work with F. Bihan [BEH15]. Consider a system

$$f(x,y) = ax + b - y = 0, \tag{1.4.1}$$

where $f \in \mathbb{R}[x,y]$, has t non-zero terms. In Chapter 3, all solutions in $(\mathbb{R}^*)^2$ are counted with multiplicities. This reduces to counting the number of real roots of a polynomial $f(x, ax+b)$, where $a, b \in \mathbb{R}$ and $f \in \mathbb{R}[x,y]$ has at most t non-zero terms. Substituting y by $ax+b$ in the polynomial f reduces the problem of computing real solutions of (1.4.1) to computing the real roots of $f(x, ax+b)$. M. Avendaño showed in [Ave09, Theorem 1.1] that (1.4.1) has at most $6t-4$ real solutions counted with multiplicities except for the possible roots 0 and $-b/a$. The question of optimality was not addressed in [Ave09] and this was the motivation for the present work. We prove the following result.

Theorem 1.7. *Let $f \in \mathbb{R}[x,y]$ be a polynomial with at most t non-zero terms and let a, b be any real numbers. Assume that the polynomial $g(x) = f(x, ax+b)$ is not identically zero. Then g has at most $6t - \mathbf{7}$ real roots counted with multiplicities except for the possible roots 0 and $-b/a$ that are counted at most once.*

The methods used in proving the latter results are elementary, and constitute a refined version of those used in [Ave09]. This might look as a small improvement of the main result of [Ave09]. In fact, this refinement is a non-trivial one, and the bound in Theorem 1.7 is optimal at least for $t = 3$.

Theorem 1.8. *The maximal number of real intersection points of a real line with a real plane curve defined by a polynomial with three non-zero terms is eleven.*

Explicitly, the real curve with equation

$$-\,0.002404\; xy^{18} + 29\; x^6y^3 + x^3y = 0 \tag{1.4.2}$$

intersects the real line $y = x + 1$ in precisely eleven points in $\mathbb{R}^2$.

The strategy to construct this example is first to deduce from the proof of Theorem 1.7 some necessary conditions on the monomials of the desired equation. Then, the use of real Grothendieck's dessins d'enfant in a novel way helps to test the feasibility of certain monomials, since manipulating this method gives a clear representation of the topology of the graph of $x \mapsto f(x, x+1)$. Ultimately, computer experimentations lead to the precise equation (1.4.2).

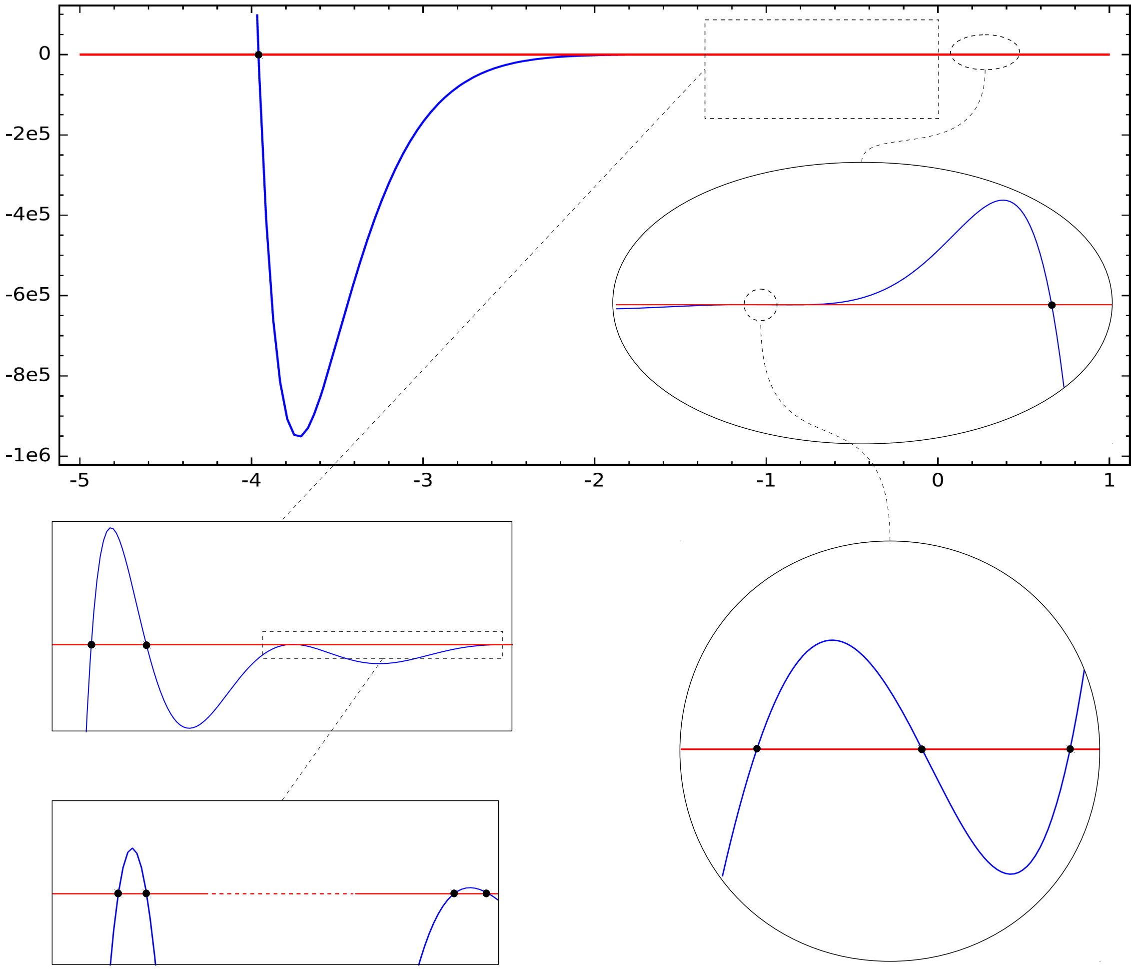

Figure 1.1: The blue curve represents the graph of $x \mapsto f(x, x+1)$, and the red line represents the first-coordinate axis. (Some parts of the curve is stretched vertically on purpose for more clarity.)

1.4.2 Chapter 4: Positive intersection points of a trinomial and a t-nomial curves

Consider a system (1.3.8) where f has $t \geq 3$ non-zero terms and g has three non-zero terms. Assume that the latter system has a finite number of solutions. Let $\mathcal{S}(3,t)$ denote the maximal number of non-degenerate positive solutions a system (1.3.8) can have. We prove the following result in Section 4.2.

Theorem 1.9. *We have* $\mathcal{S}(3,t) \leq 3 \cdot 2^{t-2} - 1$.

Note that since the number of positive solutions of two trinomials in two variables is bounded by five (see [LRW03]), the bound $\mathcal{S}(3,t)$ is sharp for $t = 3$. Moreover, for $t = 4, \ldots, 9$, this new bound is smaller than the bounds $2^t - 2$ and $2t^3/3 + 5t$, obtained in [LRW03] and [KPT15b] respectively, and shows for example that $6 \leq \mathcal{S}(3,4) \leq 11$.

Recall that substituting one variable of the trinomial g of (1.3.8) in terms of the other reduces the system to an analytic function in one variable

$$h(x) = \sum_{i=1}^{t} a_i x^{k_i} (1-x)^{l_i}.$$

The number of positive solutions of (1.3.8) is equal to that of $h = 0$ contained in $]0,1[$. We prove Theorem 1.9 using the same approach that was considered in [LRW03] i.e. we consider a recursion involving derivatives of analytic functions in one variable associated to the system (1.3.8). Beginning with the function $f_1 = h$, at each step $1 < i < t$, we are left with a function f_i defined as a certain number of derivatives of f_{i-1} multiplied by powers of x and of $(1-x)$. Using Rolle's Theorem for each f_i, one can bound the number of its roots contained in $]0,1[$ in terms of the roots of f_{i-1} in the same interval. It turns out that at the step $t-2$, we are reduced to bound the number of roots in $]0,1[$ of the equation $\phi(x) = 1$, where

$$\phi(x) = \frac{x^{\alpha}(1-x)^{\beta}P(x)}{Q(x)},$$

$\alpha, \beta \in \mathbb{Q}$, and both P and Q are real polynomials of degree at most $2^{t-2} - 1$.

The larger part of Chapter 4 is devoted to the proof in Section 4.3 of the following result.

Theorem 1.10. *We have* $\sharp\{x \in]0,1[\ |\phi(x) = 1\} \ \leq \deg P + \deg Q + 2$.

Choosing $m \in \mathbb{N}$ such that both $m\alpha$ and $m\beta$ are integers, we get a rational function $\varphi := \phi^m : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$. The inverse images of 0, 1, ∞ are given by the roots of P, Q, $\varphi - 1$, together with 0 and 1 (if $\alpha\beta \neq 0$). These inverse images lie on the graph $\Gamma := \varphi^{-1}(\mathbb{R}P^1) \subset \mathbb{C}P^1$, which is an example of a Grothendieck's real dessin d'enfant. Although this latter object Γ appears in Chapter 4 as well, we use it this time in a yet another resourceful way. In fact, there are many restrictions on the topology of the graph of φ that appear explicitly as restrictions on $\Gamma = \varphi^{-1}(\mathbb{R}P^1)$. Namely, critical points of φ correspond to vertices of Γ. The number of roots of $\varphi - 1$ in $]0,1[$ is controlled by the number of a certain type of critical points of φ called *useful positive* critical points. By doing a delicate analysis on Γ, we bound the number of vertices corresponding to these critical points in terms of $\deg P$ and $\deg Q$.

We consider in Section 4.4 the case $t = 3$ i.e. the case of two trinomials in two variables. Recall that when the maximal number of positive solutions is attained, the Minkowski sum $\Delta_1 + \Delta_2$ is an hexagon (see [LRW03]). In terms of normal fans, this means that the normal fan of the Minkowski sum $\Delta_1 + \Delta_2$, which is the common refinement of the normal fans of Δ_1 and Δ_2, has six 2-dimensional cones (and six 1-dimensional cones). We give the following additional constraints on the Minkowski sum of Δ_1 and Δ_2 when (1.3.8) has five positive solutions. We say that Δ_1 and Δ_2 alternate if every 2-dimensional cone of the normal fan of Δ_1 contains a 1-dimensional cone of the normal fan of Δ_2 having only the origin as a common face. A further analysis of Γ in the case $t = 3$ allows us to obtain the following result.

Theorem 1.11. *If the system* (1.3.8) *has 5 positive solution, then* Δ_1 *and* Δ_2 *do not alternate.*

The Newton triangles Δ_1 and Δ_2 do not alternate means that there exist two consecutive edges of $\Delta_1 + \Delta_2$ which are translate of two consecutive edges of either Δ_1 or Δ_2. Figure 7.2 illustrates this theorem for the system (7.3.6), and we provide another example in Section 4.4.

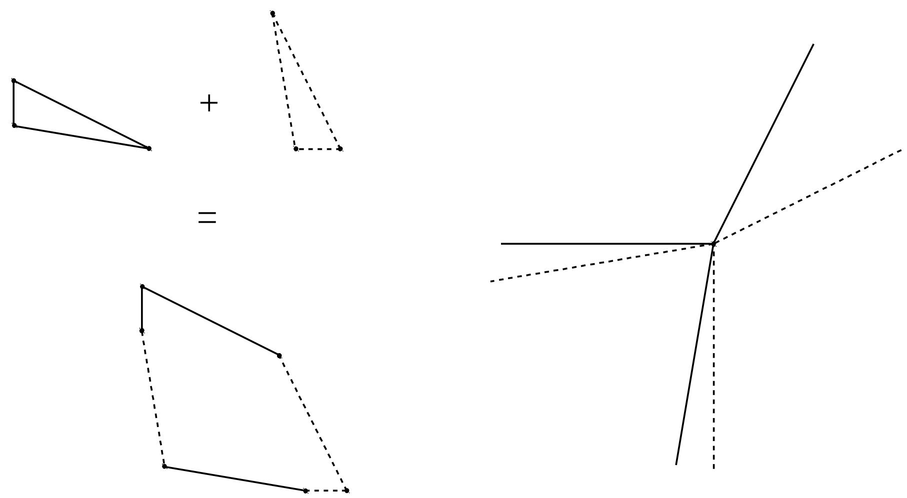

Figure 1.2: The Newton polytopes, their Minkowski sum and the associated normal fans of (7.3.6).

1.4.3 Chapter 5: Characterization of circuits supporting polynomial systems with the maximal number of positive solutions

Recall that a circuit $\mathcal{W} \subset \mathbb{R}^n$ is a set of $n+2$ distinct points that are minimally affinely dependent. A very recent generalization of Descartes' rule of sign was developed by F. Bihan and A. Dickenstein in [BD16]. This gave some conditions on both the circuit and the coefficient matrix that are necessary for the system to have $n+1$ non-degenerate positive solutions. More precisely, the authors in [BD16] show that if such a system has $n+1$ non-degenerate positive solutions, then all maximal minors of the coefficient matrix are nonzero and any affine relation $\sum_{i=1}^{n+2} \lambda_i w_i = 0$ on $\mathcal{W}$ has the same number (up to 1 if n is odd) of positive coefficients as that of negative ones. In this chapter, we completely characterize the circuits which are supports of polynomial systems with $n+1$ non-degenerate positive solutions.

Theorem 1.12. *A circuit $\mathcal{W}$ in $\mathbb{R}^n$ supports a system with $n+1$ non-degenerate positive solutions if and only if there exists a bijection*

$$\begin{array}{ccc} \{1,\ldots,n+2\} & \longrightarrow & \mathcal{W} \\ i & \longmapsto & w_i \end{array}$$

such that every affine relation on $\mathcal{W}$ can be written as

$$\sum_{i=1}^{s} \alpha_i w_i = \sum_{s+1}^{n+2} \alpha_i w_i,$$

where $s = \lfloor (n+2)/2 \rfloor$ and all α_i are positive numbers which satisfy

$$\sum_{i=1}^{r} \alpha_i < \sum_{i=s+1}^{s+r} \alpha_i < \sum_{i=1}^{r+1} \alpha_i \quad \textit{for} \quad r = 1,\ldots,s-1 \quad \textit{if} \quad n \quad \textit{is even}$$

or

$$\sum_{i=1}^{r}\alpha_i < \sum_{i=s+2}^{s+r+1}\alpha_i < \sum_{i=1}^{r+1}\alpha_i \quad \textit{for} \quad r=1,\dots,s-1 \quad \textit{if} \quad n \quad \textit{is odd.}$$

F. Bihan proved in [Bih15] that if a circuit in $\mathbb{Z}^n$ supports a maximally positive system with $n+1$ non-degenerate positive solutions, then it has a primitive affine relation (i.e. affine relation with coprime integer coefficients) as in Theorem 1.12 with $\alpha_1 = \alpha_{n+2} = 1$ and all other coefficients are equal to two. This can be seen as a consequence of Theorem 1.12 (see Example 5.12, Section 5.2). Indeed, if $\mathcal{W}$ supports a maximally positive system with $n+1$ non-degenerate positive solutions, then the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{W}$ is $\mathbb{Z}^n$. Moreover, if $\sum_{i=1}^{s}\alpha_i w_i = \sum_{s+1}^{n+2}\alpha_i w_i$ is a primitive affine relation, then $\sum_{i=1}^{s}\alpha_i = \sum_{s+1}^{n+2}\alpha_i = n+1$ (see [Bih15] for more details), which together with inequalities in Theorem 1.12 imply the desired equalities. In order to prove Theorem 1.12, one can reduce to the case where $\mathcal{W}\subset\mathbb{Z}^n$ (see the first part of Chapter 5). We prove the "only if" part of Theorem 1.12 in the following way. Consider a polynomial system supported on a circuit with n equations in n variables that has the maximal number of non-degenerate positive solutions. We associate to it using Gale duality (see Section 5.1), a univariate function

$$\varphi(y) = \prod_{i=1}^{n+1} P_i^{\lambda_i},$$

where P_i a polynomial of degree 1 that depends on the equations of the system, $\sum_{i=1}^{n+1}\lambda_i(w_i - w_0)$ is a linear relation on the vectors $w_i - w_0$ and the non-degenerate positive solutions of the initial system are in bijection with solutions of $\varphi(y) = 1$ contained in

$$\Delta_+ = \{y \in \mathbb{R}_{>0} \mid P_i(y) > 0,\ i = 1,\dots,n+1\}.$$

The homogenization of φ is a rational map $\mathbb{C}P^1 \to \mathbb{C}P^1$, so that the inverse image of $\mathbb{R}P^1$ by this homogenization is the real dessin d'enfant Γ (see Chapter 2). Since the valencies of the vertices of Γ are controlled by the integers λ_i and the roots of P_i for $i = 1,\dots,n+1$, by analysing Γ, we obtain the inequalities of Theorem 1.12.

The solutions of $\varphi(y) = 1$ in Δ_+ are roots of the *Gale polynomial*

$$G(y) = \prod_{\lambda_i>0} P_i^{\lambda_i}(y) - \prod_{\lambda_i<0} P_i^{-\lambda_i}(y) \tag{1.4.3}$$

in the same interval. In [PR13, proof of Lemma 1.8], K. Phillipson and J.-M. Rojas construct polynomial systems supported on a circuit in $\mathbb{Z}^n$ with $n+1$ non-degenerate positive solutions using *Viro polynomials* $P_{i,t}(y) = a_i + t^{\alpha_i} b_i$, where $a_i, b_i, \alpha_i \in \mathbb{R}$, and $t > 0$ is a parameter that will be taken small enough. They apply the version of Viro's combinatorial patchworking developed in [Stu94] which involves mixed subdivision of Newton polytopes. Here, we also use Viro polynomials $P_{i,t}$, and look directly at the roots of the corresponding Gale polynomial in Δ_+. The inequalities in Theorem 1.12 appear explicitly as being necessary to construct polynomial systems supported on a circuit in $\mathbb{Z}^n$ with $n+1$ non-degenerate positive solution using Viro polynomials $P_{i,t}$.

1.4.4 Chapter 6: Constructing polynomial systems with many positive solutions

Tropical geometry is a new domain in mathematics that is situated at the junction of fields such as toric geometry, complex or real geometry, and combinatorics [Mik06, MR05, MS15]. It turns out, that Sturmfels' generalization of Viro's Theorem can be reformulated in the context of tropical geometry (see [Mik04, Rul01]). This makes tropical geometry an effective tool to construct polynomial systems with prescribed support and many positive solutions.

Recall that the best known fewnomial bound on the number of non-degenerate positive solutions for a real polynomial system of n equations in n variables supported on a set of $n+k+1$ points for $k, n \geq 1$ is equal to $\frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$ [BS07]. In fact, the same paper contains the better upper bound 15 when $n = k = 2$. However, the best previously known constructions give 5 non-degenerate positive solutions (c.f. [Haa02]). The motivation behind this chapter is to implement Sturmfels' version of Viro's combinatorial patchworking and other tools and results (c.f. Chapter 2, Subsection 2.2.6) developed in tropical geometry for constructing a system of two equations in two variables and five monomials (a system of type $n = k = 2$ for short) having many positive solutions.

Let $\mathbb{K}$ be the field of **generalized locally convergent Puiseux series**

$$a(t) = \sum_{r \in R} \alpha_r t^r,$$

where $R \subset \mathbb{R}$ is a well ordered set and $a(t)$ is a complex series convergent for $t > 0$ small enough. This is an algebraically closed field. Consider the subfield $\mathbb{RK}$ of $\mathbb{K}$ of *real* generalized Puiseux series, that is all α_r appearing in $a(t)$ are real numbers. We consider in this chapter a sparse (Laurent) polynomial system

$$f_1(z) = f_2(z) = 0, \tag{1.4.4}$$

with equations defined over $\mathbb{RK}$. We assume that (1.4.4) has finitely many solutions, and all of them are non-degenerate. A **positive** element $a(t)$ of $\mathbb{K}$ is an element of $\mathbb{RK}^*$ whose first-order term has positive coefficient.

To a Laurent polynomial $f(z) = \sum_{w \in \mathcal{W}} c_w z^w \in \mathbb{R}[z]$, one associates a *tropical polynomial*

$$f_{\text{trop}}(x) = \text{``} \sum_{w \in \mathcal{W}} \text{val}(c_w) x^w \text{''},$$

where $\text{val}(c_w)$ is minus the order (in the classical sense) of the Puiseux series c_w, and the operations are the tropical ones (the sum is the max, and the product is the classical sum). The associated *tropical hypersurface* T is the corner locus of the piecewise-linear convex function $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f_{\text{trop}}(x)$. By Kapranov's Theorem [Kap00] (see Subsection 2.2.2), the tropical hypersurface T coincides with the closure of

$$\text{Val}\left(\{z \in (\mathbb{K}^*)^n \mid f(z) = 0\}\right),$$

where Val is the extension of the function val coordinate-wise. The **positive part** of T is the closure of $\text{Val}\left(\{z \in (\mathbb{RK}_{>0})^n \mid f(z) = 0\}\right)$.

Consider now again polynomials $f_1, f_2 \in \mathbb{RK}[z_1^{\pm 1}, z_2^{\pm 1}]$ defining two tropical curves $T_1, T_2 \subset \mathbb{R}^2$. Assume for the moment that T_1 and T_2 intersect transversally, which means that each intersection point is isolated and contained in the relative interiors of one 1-dimensional linear piece of T_1 and one 1-dimensional linear piece of T_2. Then by Sturmfels' generalization of Viro's theorem, each intersection point of T_1 and T_2 contained in both positive parts (positive intersection point for short) lifts to a unique solution of (1.4.4) in $(\mathbb{RK}_{>0})^2$, which gives a positive solution of a real system $g_1(z) = g_2(z) = 0$ by taking $t > 0$ small enough. Recall that in the case $n = k = 2$ (meaning that equations of T_1 and T_2 have a total of five monomials), the number of transversal intersection points of T_1 and T_2 is bounded from above by six (see Subsection 1.3.2). We prove that this bound is sharp and can be realized by positive intersection points.

Proposition 1.13. *There exist two plane tropical curves T_1 and T_2 defined by equations containing a total of five monomials and which have six positive transversal intersection points.*

Therefore, using Sturmfels' generalization of Viro's theorem (as explained above), this gives a real system of type $n = k = 2$ having six non-degenerate positive solutions. In order to get a real system of type $n = k = 2$ with more than six non-degenerate positive solutions, we thus consider tropical curves T_1 and T_2 which do not intersect transversally.

Note that $T_1 \cap T_2$ is piecewice-linear and its linear pieces are either isolated points or line segments. Luckily, if a linear piece $\xi \subset T_1 \cap T_2$ is an isolated point, then results in [Kat09, Rab12, OP13] and [BLdM12] show that ξ lifts to a solution of (1.4.4) in $(\mathbb{K}^*)^2$, and then non-degenerate positive solutions of (1.4.4) with valuation equal to ξ can be estimated by computing the real *reduced system* of (1.4.4) with respect to ξ (see Chapter 2, Subsection 2.2.6). However, if such a linear piece ξ has dimension 1, then ξ is an infinite set containing a finite (and possibly empty) set of points that are valuations of non-degenerate positive solutions of (1.4.4). Locating such valuations does not come easily. In fact, there is only one known method for achieving this, called *tropical modification* (see [Mik06, BLdM12]). This problem is addressed in Section 6.2 of Chapter 6 using another approach. Namely, for each linear piece ξ of dimension 1, we associate a univariate Viro polynomial $f_{t,\xi}$ so that all the first-order terms of non-degenerate positive solutions of (1.4.4) with valuations in the relative interior of ξ can be recovered from both the reduced system of (1.4.4) with respect ξ, and the Viro polynomial $f_{t,\xi}$.

We now consider a system (1.4.4) of type $n = k = 2$. Assume that no three points of the support of the system belong to a line. We prove in Section 6.3 that one can associate to such a system a new system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0, \end{aligned} \tag{1.4.5}$$

with polynomials in $\mathbb{R}\mathbb{K}[y_1^{\pm 1}, y_2^{\pm 1}]$, that has the same number of positive non-degenerate solutions as (1.4.4), and satisfying that all a_i, b_j have zero order, all m_i, n_i belong to $\mathbb{Z}$ with $m_1, n_2 > 0$, and both α, β are real numbers.

The two main results of Chapter 6 are the following.

Theorem 1.14. *If $(\alpha, \beta) \neq (0, 0)$, then (1.4.5) has at most nine non-degenerate positive solutions.*

We prove Theorem 1.14 in Section 6.5. Note that if $(\alpha, \beta) = (0, 0)$, then there is nothing that can be done using tropical geometry. Indeed, the task of bounding the number of non-degenerate positive solutions of (1.4.5) becomes equivalent to computing the number of positive solutions of a real polynomial system of type $n = k = 2$.

Theorem 1.15. *There exists a system (1.4.5) that has seven non-degenerate positive solutions.*

The construction of a system (1.4.5) that has seven non-degenerate positive solutions is made in Section 6.5. Namely, for any $0 < \alpha < \gamma_0$, the system

$$\begin{aligned} -1 + y_1^6 + y_1^3 y_2^6 - t^{\alpha} y_1^{-14} y_2^7 &= 0, \\ -1 + 0.36008 t^{\gamma_0} + y_1^6 + (1 - 0.36008 t^{\alpha}) y_1^3 y_2^6 - (44/31)^{\frac{5}{6}} t^{\alpha} y_1^{-12} y_2^9 &= 0, \end{aligned} \tag{1.4.6}$$

has seven non-degenerate positive solutions.

We made a tedious case-by-case analysis to get necessary conditions for (1.4.5) to have more than six non-degenerate positive solutions. As a by-product, we obtain in Sections 6.6 and 6.7 the following result.

Theorem 1.16. *If* $(\alpha, \beta) \neq (0, 0)$*, and one of the following is true*

1. *For* $i = 0, 2$*, the coefficient of the first order term of* a_i *is different from that of* b_i*,*
2. $\alpha \neq \beta$*,*
3. $\alpha = \beta < 0$*,*

then (1.4.5) *has at most six non-degenerate positive solutions.*

Chapter 2

Preliminaries

2.1 A brief introduction to real dessins d'enfant

For more details, see [Ore03, Bru06, Bih07] for example. Consider a real rational map $\varphi = \dfrac{P}{Q} : \mathbb{C} \to \mathbb{C}$, where P and Q are two real polynomials. The degree of φ is the maximum of the degrees of P and Q. We extend φ to a rational homogeneous function $\mathbb{C}P^1 \to \mathbb{C}P^1$, $(x_0 : x_1) \mapsto (1 : P/Q)$, that we denote again by φ. Define

$$\Gamma := \varphi^{-1}(\mathbb{R}P^1).$$

This is a real graph on $\mathbb{C}P^1$ invariant with respect to the complex conjugation and which contains $\mathbb{R}P^1$. Any connected component of $\mathbb{C}P^1 \setminus \Gamma$ is homeomorphic to an open disk. Moreover, each vertex of Γ has even valency, and the multiplicity of a critical point with real critical value of φ is half its valency. The graph Γ contains the inverse images of $(1 : 0)$, $(0 : 1)$ and $(1 : 1)$, which are the sets of roots of P, Q and $P/Q - 1$ respectively. Denote by the same letter p (resp. q and r) the points of Γ which are mapped to $(1 : 0)$ (resp. $(0 : 1)$ and $(1 : 1)$). Orient the real axis on the target space via the arrows $0 \to \infty \to 1 \to 0$ (orientation given by the decreasing order in $\mathbb{R}$), which is equivalent to orienting $\mathbb{R}P^1$ via the arrows $(1 : 0) \to (0 : 1) \to (1 : 1)$. Pull back this orientation by φ, the graph Γ becomes an oriented graph, with the orientation given by arrows $p \to q \to r \to p$. A **cycle** of Γ is the boundary of a connected component of $\mathbb{C}P^1 \setminus \Gamma$. Any such cycle contains the same non-zero number of letters r, p , q (see Figure 2.1). We say that a cycle obeys the **cycle rule**. The graph Γ is called *real dessin d'enfant* associated to φ. Since Γ is invariant under complex conjugation, it is determined by its intersection with one connected component H (for half) of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. Since φ is real, its degree is the sum of the degrees of its restrictions to connected components of $\mathbb{C}P^1 \setminus \Gamma$. To represent the real dessin d'enfant, we draw a horizontal line corresponding to the real projective line and draw below one half $H\Gamma$ of Γ, see Figure 3.1 for instance.

Clearly, the arrangement of real roots of P, Q and $P/Q - 1$ together with their multiplicities can be extracted from the graph Γ. We encode this arrangement together with the multiplicities by what is called a root scheme.

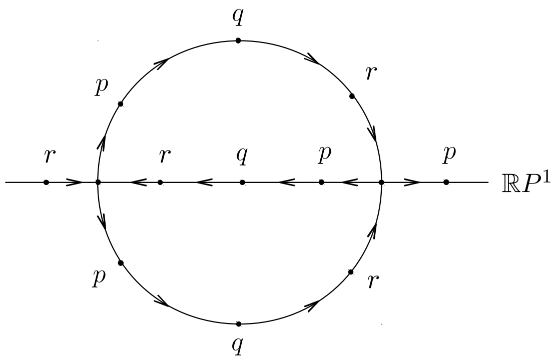

Figure 2.1: Cycles of Γ obeying the cycle rule.

Definition 2.1 ([Bru06, Ore03]). *A* ***root scheme*** *is a k-tuple $(l_1, m_1), \ldots, (l_k, m_k) \in (\{p, q, r\} \times \mathbb{N})^k$. A root scheme is realizable by polynomials of degree d if there exist real polynomials P and Q such that φ has degree d and if $x_1 < \ldots < x_k$ are the real roots of P, Q and $P/Q - 1$, then $l_i = p$ (resp. q, r) if x_i is a root of P (resp. Q, $P/Q - 1$) and m_i is the multiplicity of x_i.*

Conversely, suppose we are given a real graph $\Gamma \subset \mathbb{C}P^1$ that is invariant under complex conjugation, together with a real continuous map $\phi : \Gamma \to \mathbb{R}P^1$. Denote the inverse images of 0, ∞ and 1 by letters p, q and r, respectively, and orient Γ with the pull back by ϕ of the above orientation of $\mathbb{R}P^1$. This graph is called a *real rational graph* [Bru06] if any vertex of Γ has even valency and any connected component of $\mathbb{C}P^1 \setminus \Gamma$ is homeomorphic to an open disk. Then, for any connected component D of $\mathbb{C}P^1 \setminus \Gamma$, the map $\phi_{|\partial D}$ is a covering of $\mathbb{R}P^1$ whose degree d_D is the number of letters p (resp. q, r) in ∂D. We define the degree of Γ to be half the sum of the degrees d_D over all connected components of $\mathbb{C}P^1 \setminus \Gamma$. Since ϕ is a real map, the degree of Γ is also the sum of the degrees d_D over all connected components D of $\mathbb{C}P^1 \setminus \Gamma$ contained in one connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$.

The following result [Ore03] explains the importance of real rational graphs in computing the roots of $P/Q - 1$.

Proposition 2.2 (Orevkov). *A root scheme is realizable by polynomials of degree d if and only if it can be extracted from a real rational graph of degree d on $\mathbb{C}P^1$.*

We show now how to prove the if part in Proposition 2.2 (see [Bih07, Bru06, Ore03]). For each connected component D of $\mathbb{C}P^1 \setminus \Gamma$, extend $\phi_{|\partial D}$ to a branched covering of degree d_D (use the map $z \mapsto z^{d_D}$) of one connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$, so that two adjacent connected components of $\mathbb{C}P^1 \setminus \Gamma$ project to different connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. Then, it is possible to glue continuously these maps in order to obtain a real branched covering $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree d. The map ϕ becomes a real rational map of degree d for the standard complex structure on the target space and its pull-back by ϕ on the source space. There exist then real polynomials P and Q such that P/Q has degree d and $\phi = P/Q$, so that the points p (resp. q, r) correspond to the roots of P (resp. Q, $P/Q - 1$) and $\Gamma = \phi^{-1}(\mathbb{R}P^1)$.

2.2 A brief introduction to tropical geometry

The notations in this section are taken from [BLdM12, BB13, Ren15, GL15].

2.2.1 Polytopes and subdivisions

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, endowed with the standard inner product $\langle \ , \ \rangle : \ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Definition 2.3. *A* ***rational polyhedron*** *in $\mathbb{R}^n$ is a convex set of points x, defined by a finite number of inequalities of type*

$$\langle x, w \rangle \leq c,$$

where $w \in \mathbb{Z}^n$ and $c \ \in \mathbb{R}^n$.

If a rational polyhedron is closed, then it is called an **integer convex polytope**. All polytopes considered in Chapter 6 are integer convex.

Definition 2.4. *A* ***rational polyhedral complex*** *is a finite set of rational polyhedra $\mathcal{P} = \{\Delta_i\}_i$ such that*

1. *for every $\Delta \in \mathcal{P}$, if Δ' is a face of Δ, then $\Delta' \in \mathcal{P}$, and*

2. *if $\Delta, \Delta' \in \mathcal{P}$, then $\Delta \cap \Delta'$ is a face of both Δ and Δ'.*

Let F be a field of characteristic zero. For $z = (z_1, \ldots, z_n) \in F^n$ and $w = (w^1, \ldots, w^n) \in \mathbb{R}^n$, set $z^w = z_1^{w^1} \cdots z_n^{w^n}$. Consider a polynomial $f = \sum_{w \in \mathcal{W}} c_w z^w \ \in F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, with $\mathcal{W} \neq \emptyset$ a finite subset of $\mathbb{Z}^n$, and $c_w \in F^*$.

Definition 2.5. *The* ***Newton polytope*** *$\Delta(f)$ of f is defined to be the convex hull* $\mathrm{Conv}(\mathcal{W})$ *of $\mathcal{W}$.*

Definition 2.6. *A* ***polyhedral subdivision*** *of an integer convex polytope Δ is a set of integer convex polytopes $\{\Delta_i\}_{i \in I}$ such that*

- *$\cup_{i \in I} \Delta_i = \Delta$, and*

- *if $i, j \ \in I$, then if the intersection $\Delta_i \cap \Delta_j$ is non-empty, it is a common face of the polytope Δ_i and the polytope Δ_j.*

Definition 2.7. *Let Δ be an integer convex polytope in $\mathbb{R}^n$ and let τ denote a polyhedral subdivision of Δ consisting of integer convex polytopes. We say that τ is* ***regular*** *if there exists a continuous, convex, piecewise-linear function $\varphi : \ \Delta \to \mathbb{R}$ which is affine linear on every simplex of τ.*

Let Δ be an integer convex polytope in $\mathbb{R}^n$ and let $\phi : \ \Delta \cap \mathbb{Z}^n \to \mathbb{R}$ be a function. We denote by $\hat{\Delta}(\phi)$ the convex hull of the graph of ϕ, i.e.,

$$\hat{\Delta}(\phi) := \mathrm{Conv}\left(\{ (i, \phi(i)) \ \in \mathbb{R}^{n+1} \mid i \in \Delta \cap \mathbb{Z}^n \} \right).$$

Then the polyhedral subdivision of Δ, induced by projecting the union of the lower faces of $\hat{\Delta}(\phi)$ onto the first n coordinates, is regular. In the following, we describe how we define ϕ using the polynomials that we will be working with.

2.2.2 Tropical polynomials and hypersurfaces

A **locally convergent generalized Puiseux series** is a formal series of the form

$$a(t) = \sum_{r\in R} \alpha_r t^r,$$

where $R \subset \mathbb{R}$ is a well-ordered set, all $\alpha_r \in \mathbb{C}$, and the series is convergent for $t > 0$ small enough. We denote by $\mathbb{K}$ the set of all locally convergent generalized Puiseux series. It is naturally a field of characteristic 0, which turns out to be algebraically closed.

Notation 2.8. *Let* $\operatorname{coef}(a(t))$ *denote the coefficient of the first term of* $a(t)$ *following the increasing order of the exponents of* t. *We extend* coef *to a map* $\operatorname{Coef}: \mathbb{K}^n \to \mathbb{R}^n$ *by taking* coef *coordinate-wise, i.e.* $\operatorname{Coef}(a_1(t), \dots, a_n(t)) = (\operatorname{coef}(a_1(t)), \dots, \operatorname{coef}(a_n(t)))$

An element $a(t) = \sum_{r\in R} \alpha_r t^r$ of $\mathbb{K}$ is said to be **real** if $\alpha_r \in \mathbb{R}$ for all r, and **positive** if $a(t)$ is real and $\operatorname{coef}(a(t)) > 0$.

Denote by $\mathbb{RK}$ (resp. $\mathbb{RK}_{>0}$) the subfield of $\mathbb{K}$ composed of real (resp. positive) series. Since elements of $\mathbb{K}$ are convergent for $t > 0$ small enough, an algebraic variety over $\mathbb{K}$ (resp. $\mathbb{RK}$) can be seen as a one parametric family of algebraic varieties over $\mathbb{C}$ (resp. $\mathbb{R}$). The field $\mathbb{K}$ has a natural non-archimedian valuation defined as follows:

$$\begin{array}{rccc} \operatorname{val}: & \mathbb{K} & \longrightarrow & \mathbb{R}\cup\{-\infty\} \\ & 0 & \longmapsto & -\infty \\ & \sum_{r\in R} \alpha_r t^r \neq 0 & \longmapsto & -\min_R\{r \mid \alpha_r \neq 0\}. \end{array}$$

The valuation extends naturally to a map $\operatorname{Val}: \mathbb{K}^n \to (\mathbb{R}\cup\{-\infty\})^n$ by evaluating val coordinate-wise, i.e. $\operatorname{Val}(z_1, \dots, z_n) = (\operatorname{val}(z_1), \dots, \operatorname{val}(z_n))$. We shall often use the notation val and Val when the context is a *tropical polynomial* or a *tropical hypersurface*. On the other hand, define $\operatorname{ord} := -\operatorname{val}$, with $\operatorname{ord}(0) = +\infty$, and use it as a notation when the context is an element in $\mathbb{RK}^n$ or a polynomial in $\mathbb{RK}[z_1^{\pm 1}, \dots, z_2^{\pm 1}]$.

Convention 2.9. *For any* $s \in \mathbb{K}$, *we have* $\operatorname{coef}(s) = 0 \Leftrightarrow s = 0$ *and* $\operatorname{ord}(s) = +\infty \Leftrightarrow s = 0$

Consider a polynomial

$$f(z) := \sum_{w\in\mathcal{W}} c_w z^w \in \mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}],$$

with $\mathcal{W}$ a finite subset of $\mathbb{Z}^n$ and all c_w are non-zero. Let $V_f = \{z \in (\mathbb{K}^*)^2 \mid f(z) = 0\}$ be the zero set of f in $(\mathbb{K}^*)^n$

The **tropical hypersurface** V_f^{trop} associated to f is the closure (in the usual topology) of the image under Val of V_f:

$$V_f^{\text{trop}} = \overline{\operatorname{Val}(V_f)} \subset \mathbb{R}^n,$$

endowed with a *weight function* which we will define later. There are other equivalent definitions of a tropical hypersurface. Namely, define

$$\begin{array}{rccc} \nu: & \mathcal{W} & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \operatorname{ord}(c_w). \end{array}$$

Its **Legendre transform** is a piecewise-linear convex function

$$\begin{array}{rccc} \mathcal{L}(\nu): & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \max_{w\in\mathcal{W}}\{\langle x,w\rangle - \nu(w)\}. \end{array}$$

We have the fundamental Theorem of Kapranov [Kap00].

Theorem 2.10 (Kapranov)**.** *A tropical hypersurface* V_f^{trop} *is the corner locus of* $\mathcal{L}(\nu)$*.*

The corner locus of $\mathcal{L}(\nu)$ is the set of points at which it is not differentiable. Tropical hypersurfaces can also be described as algebraic varieties over the *tropical semifield* $(\mathbb{R}\cup\{-\infty\}, \text{“}+\text{”}, \text{“}\times\text{”})$, where for any two elements x and y in $\mathbb{R}\cup\{-\infty\}$, one has

$$\text{“}x+y\text{”} = \max(x,y) \quad \text{and} \quad \text{“}x\times y\text{”} = x+y.$$

A multivariate tropical polynomial is a polynomial in $\mathbb{R}[x_1,\ldots,x_n]$, where the addition and multiplication are the tropical ones. Hence, a tropical polynomial is given by a maximum of finitely many affine functions whose linear parts have integer coefficients and constant parts are real numbers. The tropicalization of the polynomial f is a tropical polynomial

$$f_{\text{trop}}(x) = \max_{w\in\mathcal{W}}\{\langle x,w\rangle + \text{val}(c_w)\}.$$

This tropical polynomial coincides with the piecewise-linear convex function $\mathcal{L}(\nu)$ defined above. Therefore, Theorem 2.10 asserts that V_f^{trop} is the corner locus of f_{trop}. Conversely, the corner locus of any tropical polynomial is a tropical hypersurface.

2.2.3 Tropical hypersurfaces and subdivisions

A tropical hypersurface induces a subdivision of the Newton polytope $\Delta(f)$ in the following way. The hypersurface V_f^{trop} is a $(n-1)$-dimensional piecewise-linear complex which induces a polyhedral subdivision Ξ of $\mathbb{R}^n$. We will call **cells** the elements of Ξ. Note that these cells have rational slopes. The n-dimensional cells of Ξ are the closures of the connected components of the complement of V_f^{trop} in $\mathbb{R}^n$. The lower dimensional cells of Ξ are contained in V_f^{trop} and we will just say that they are cells of V_f^{trop}.

Consider a cell ξ of V_f^{trop} and pick a point x in the relative interior of ξ. Then the set

$$\mathcal{I}_x = \{w\in\Delta(f)\cap\mathbb{Z}^n \mid \exists\, x\in\mathbb{R}^n,\ f_{trop}(x) = \langle x,w\rangle + \text{val}(c_w)\}$$

is independent of x, and denote by Δ_ξ the convex hull of this set. All together the polyhedra Δ_ξ form a subdivision τ of $\Delta(f)$ called the **dual subdivision**, and the cell Δ_ξ is called the **dual** of ξ. Both subdivisions τ and Ξ are dual in the following sense. There is a one-to-one correspondence between Ξ and τ, which reverses the inclusion relations, and such that if $\Delta_\xi\in\tau$ corresponds to $\xi\in\Xi$ then

1. $\dim\xi + \dim\Delta_\xi = n$,
2. the cell ξ and the polytope Δ_ξ span orthogonal real affine spaces,
3. the cell ξ is unbounded if and only if Δ_ξ lies on a proper face of $\Delta(f)$.

Note that τ coincides with the regular subdivision of Definition 2.7 described in Subsection 2.2.1. Indeed, let $\hat{\Delta}(f) \subset \mathbb{R}^n \times \mathbb{R}$ be the convex hull of the points $(w, \nu(w))$ with $w \in \mathcal{W}$ and $\nu(w) = \operatorname{ord}(c_w)$. Define

$$\begin{array}{rccc} \hat{\nu}: & \Delta(f) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \min\{y \mid (x,y) \in \hat{\Delta}(f)\}. \end{array}$$

Then, the the domains of linearity of $\hat{\nu}$ form the dual subdivision τ.

Consider a facet (face of dimension $n-1$) ξ of V_f^{trop}, then $\dim \Delta_\xi = 1$ and we define the **weight** of ξ by $w(\xi) := Card(\Delta_\xi \cap \mathbb{Z}^n) - 1$. Tropical varieties satisfy the so-called balancing condition. Since in Chapter 6, we only work with tropical curves in $\mathbb{R}^2$, we give here this property only for this case. We refer to [Mik06] for the general case. Let $T \subset \mathbb{R}^n$ be a tropical curve, and let v be a vertex of T. Let $\xi_1, \ldots, \xi_l$ be the edges of T adjacent to v. Since T is a rational graph, each edge ξ_i has a primitive integer direction. If in addition we ask that the orientation of ξ_i defined by this vector points away from v, then this primitive integer vector is unique. Let us denote by $u_{v,i}$ this vector.

Proposition 2.11 (Balancing condition)**.** *For any vertex v, one has*

$$\sum_{i=1} w(\xi_i) u_{v,i} = 0.$$

2.2.4 Intersection of tropical hypersurfaces

Consider polynomials $f_1, \ldots, f_k \in \mathbb{K}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. For $i = 1, \ldots, k$, let $\Delta_i \subset \mathbb{R}^n$ (resp. $T_i \subset \mathbb{R}^n$) denote the Newton polytope (resp. tropical curve) associated to f_i. Recall that each tropical curve T_i defines a piecewise linear polyhedral subdivision Ξ_i of $\mathbb{R}^n$ which is dual to a convex polyhedral subdivision τ_i of Δ_i. The union of these tropical curves defines a piecewise-linear polyhedral subdivision Ξ of $\mathbb{R}^n$. Any non-empty cell of Ξ can be written as

$$\xi = \xi_1 \cap \cdots \cap \xi_k$$

with $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$. We require that ξ does not lie in the boundary of any ξ_i, thus any cell $\xi \in \Xi$ can be uniquely written in this way. Denote by τ the mixed subdivision of the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_k$ induced by the tropical polynomials $f_1, \ldots, f_k$. Recall that any polytope $\sigma \in \tau$ comes with a privileged representation $\sigma = \sigma_1 + \cdots + \sigma_k$ with $\sigma_i \in \tau_i$ for $i = 1, \ldots, k$. The above duality-correspondence applied to the (tropical) product of the tropical polynomials gives rise to the following well-known fact (see [BB13] for instance).

Proposition 2.12. *There is a one-to-one duality correspondence between Ξ and τ, which reverses the inclusion relations, and such that if $\sigma \in \tau$ corresponds to $\xi \in \Xi$, then*

1. *if $\xi = \xi_1 \cap \cdots \cap \xi_k$ with $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$, then σ has representation $\sigma = \sigma_1 + \cdots + \sigma_k$ where each σ_i is the polytope dual to ξ_i.*

2. $\dim \xi + \dim \sigma = n$,

3. *the cell ξ and the polytope σ span orthonogonal real affine spaces,*

4. *the cell ξ is unbounded if and only if σ lies on a proper face of Δ.*

Notation 2.13. *In what follows, we denote such a* σ *by* Δ_ξ *and we say that each polytope* Δ_ξ *a* ***mixed polytope*** *of* τ.

Definition 2.14. *A cell* ξ *is* ***transversal*** *if it satisfies* $\dim(\Delta_\xi) = \dim(\Delta_{\xi_1}) + \cdots + \dim(\Delta_{\xi_k})$, *and it is* ***non transversal*** *if the previous equality does not hold.*

2.2.5 Generalized Viro theorem and tropical reformulation

An important direction in real algebraic geometry is the construction of real algebraic hypersurfaces with prescribed topology (see [Ris92, Vir84] or [Vir89] for example). Central to these developments is a combinatorial construction due to O.Ya. Viro, which is based on regular triangulations of Newton polytopes. Using this technique, significant progress has been made in the study of low degree curves in the real projective plane (Hilbert's 16th problem). Since Chapter 6 of this thesis concerns algebraic sets of dimension zero contained in $(\mathbb{R}_{>0})^n$, we only describe in this section how to use *combinatorial patchworking* in that orthant of $\mathbb{R}^n$.

Following the description of B. Sturmfels [Stu94], we recall now Viro's Theorem for hypersurfaces. Let $\mathcal{W} \subset \mathbb{Z}^n$ be a finite set of lattice points, and denote by Δ the convex hull of $\mathcal{W}$. Assume that $\dim \Delta = n$ and let $\varphi : \mathcal{W} \to \mathbb{Z}$ be any function inducing a regular triangulation τ_φ of the integer convex polytope Δ (see Definition 2.7). Fix non-zero real numbers c_w, $w \in \mathcal{W}$. For each positive real number t, we consider a Laurent polynomial

$$f_t(z_1, \ldots, z_n) = \sum_{w \in \mathcal{W}} c_w t^{\varphi(w)} z^w. \tag{2.2.1}$$

Let $\mathrm{Bar}(\tau_\varphi)$ denote the first barycentric subdivision of the regular triangulation τ_φ. Each maximal cell μ of $\mathrm{Bar}(\tau_\varphi)$ is incident to a unique point $w \in \mathcal{W}$. We define the sign of a maximal cell μ to be the sign of the associated real number c_w. The sign of any lower dimensional cell $\lambda \in \mathrm{Bar}(\tau_\varphi)$ is defined as follows:

$$\mathrm{sign}(\lambda) := \begin{cases} + & \text{if} \quad \mathrm{sign}(\mu) = + \quad \text{for all maximal cells} \quad \mu \quad \text{containing } \lambda, \\ - & \text{if} \quad \mathrm{sign}(\mu) = - \quad \text{for all maximal cells} \quad \mu \quad \text{containing } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{Z}_+(\tau_\varphi, f)$ denote the subcomplex of $\mathrm{Bar}(\tau_\varphi)$ consisting of all cells λ with $\mathrm{sign}(\lambda) = 0$, and let $V_+(f_t)$ denote the zero set of f_t in the positive orthant of $\mathbb{R}^n$. Denote by $\mathrm{Int}(\Delta)$ the relative interior of Δ.

Theorem 2.15 (Viro)**.** *For sufficiently small* $t > 0$*, there exists a homeomorphism* $(\mathbb{R}_{>0})^n \to \mathrm{Int}(\Delta)$ *sending the real algebraic set* $V_+(f_t) \subset (\mathbb{R}_{>0})^n$ *to the simplicial complex* $\mathcal{Z}_+(\tau_\varphi, f) \subset \mathrm{Int}(\Delta)$.

Naturally, a signed version of Theorem 2.15 holds in each of the 2^n orthants

$$(\mathbb{R}_{>0})^\epsilon := \{(x_1, \ldots, x_n) \in (\mathbb{R}^*)^n \mid \mathrm{sign}(x_i) = \epsilon_i \text{ for } i = 1, \ldots, n\},$$

where $\epsilon \in \{+, -\}^n$. In fact, O. Viro proves a more general Theorem for Theorem 2.15, in which he defines a set that is homeomorphic to the the zero set $V(f_t) \subset \mathbb{R}^n$ (not only the positive zero set $V_+(f_t)$) by means of gluing the zero sets of f_t contained in all other orthants of $\mathbb{R}^n$.

We now reformulate Theorem 2.15 using tropical geometry. We consider $g := f_t$ as a polynomial defined over the field of real generalized locally convergent Puiseux series, where each coefficient

$c_w t^{\varphi(w)} \in \mathbb{RK}^*$ of g has only one term. Therefore $\mathrm{coef}(c_w t^{\varphi(w)}) = c_w$, $\mathrm{val}(c_w t^{\varphi(w)}) = -\varphi(w)$, and we associate to g a tropical hypersurface V_g^{trop} as defined in Subsection 2.2.2. Recall that V_g^{trop} induces a subdivision Ξ_g of $\mathbb{R}^n$ that is dual to τ_φ. The tropical hypersurface V_g^{trop} is homeomorphic to the barycentric subdivision $\mathrm{Bar}(\tau_\varphi)$. Indeed, τ_φ is a triangulation, and thus $\mathrm{Bar}(\tau_\varphi)$ becomes dual to τ_φ in the sense of the duality described in Subsection 2.2.3.

We define for each n-cell $\xi \in \Xi_g$, dual to a 0-face (vertex) w of the triangulation τ_φ, a sign $\epsilon(w) \in \{+, -\}$, to be equal to the sign of c_w.

Definition 2.16. *The* ***positive part****, denoted by $V_{g,+}^{\mathrm{trop}}$, is the subcomplex of V_g^{trop} consisting of all $(n-1)$-cells of V_g^{trop} that are adjacent to two n-cells of V_g^{trop} having different signs. A* ***positive facet*** *ξ_+ is an $(n-1)$-dimensional cell of $V_{g,+}^{\mathrm{trop}}$.*

The following is a Corollary of Mikhalkin [Mik04] and Rullgard [Rul01] results, where they completely describe the topology of $V(f_t)$ using *amoebas*.

Theorem 2.17 (Mikhalkin, Rullgard)**.** *For sufficiently small $t > 0$, there exists a homeomorphism $(\mathbb{R}_{>0})^n \to \mathbb{R}^n$ sending the zero set $V_+(f_t) \subset (\mathbb{R}_{>0})^n$ to $V_{g,+}^{\mathrm{trop}} \subset \mathbb{R}^n$.*

B. Sturmfels generalized Viro's method for complete intersections in [Stu94]. We give now a tropical reformulation of one of the main Theorems of [Stu94].

Consider a system

$$f_{1,t}(z_1, \ldots, z_n) = \cdots = f_{k,t}(z_1, \ldots, z_n) = 0, \tag{2.2.2}$$

of k equations, where all $f_{t,i}$ are polynomial (2.2.1). For $i = 1, \ldots, k$, we define as before $g_i := f_{i,t}$ as a polynomial in $\mathbb{RK}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Let $V_+(f_{1,t}, \ldots, f_{k,t}) \subset (\mathbb{R}_{>0})^n$ denote the set of positive solutions of (2.2.2).

Theorem 2.18 (Sturmfels)**.** *Assume that the tropical hypersurfaces $V_{g_1}^{\mathrm{trop}}, \ldots, V_{g_k}^{\mathrm{trop}}$ intersect transversally. Then for sufficiently small $t > 0$, there exists a homeomorphism $(\mathbb{R}_{>0})^n \to \mathbb{R}^n$ sending the real algebraic set $\mathcal{Z}_+(f_{1,t}, \ldots, f_{k,t}) \subset (\mathbb{R}_{>0})^n$ to the intersection $V_{g_1,+}^{\mathrm{trop}} \cap \cdots \cap V_{g_k,+}^{\mathrm{trop}} \subset \mathbb{R}^n$.*

Similarly to O. Viro's work, B. Sturmfels generalizes Theorem 2.18 for the zero set $V(f_{1,t}, \ldots, f_{k,t}) \subset \mathbb{R}^n$ (see [Stu94, Theorem 5]).

2.2.5.1 Transversal intersection points and discrete mixed volume

Assume now that the number of polynomials in (2.2.2) is equal to that of variables (i.e. $k = n$), and assume that the tropical hypersurfaces $V_{g_1}^{\mathrm{trop}}, \ldots, V_{g_n}^{\mathrm{trop}}$ intersect transversally. Then the intersection set $V_+^{\mathrm{trop}}(g_1, \ldots, g_n) := V_{g_i,+}^{\mathrm{trop}} \cap \cdots \cap V_{g_k,+}^{\mathrm{trop}}$ is a (possibly empty) set of points in $\mathbb{R}^n$. Each point p of $V_+^{\mathrm{trop}}(g_1, \ldots, g_n)$ is expressed in a unique way as a transversal intersection $\xi_{1,+} \cap \cdots \cap \xi_{n,+}$, where for $i = 1, \ldots, n$, the cell $\xi_{i,+} \subset V_{g_i,+}^{\mathrm{trop}}$ is a positive cell. Theorem 2.18 is a powerful tool for constructing polynomial systems (2.2.2) with many non-degenerate positive solutions.

A consequence of F. Bihan's more general result [Bih14] is a bound on the number of positive mixed points for a system (2.2.2). For any number r of finite sets $\mathcal{W}_1, \ldots, \mathcal{W}_r$ in $\mathbb{R}^n$, and for any non-empty $I \subset [r] = \{1, 2, \ldots, r\}$, write $\mathcal{W}_I$ for the set of points $\sum_{i \in I} w_i$ over all $w_i \in \mathcal{W}_i$ with $i \in I$. The associated *discrete mixed volume* of $\mathcal{W}_1, \ldots, \mathcal{W}_r$ is defined as

$$D(\mathcal{W}_1, \ldots, \mathcal{W}_r) = \sum_{I \subset [r]} (-1)^{r-|I|} |\mathcal{W}_I|, \tag{2.2.3}$$

where the sum is taken over all subsets I of $[r]$ including the empty set with the convention that $|\mathcal{W}_\emptyset| = 1$. Denote by $\mathcal{W}_i$ the support of g_i for $i = 1, \dots, n$. Recall that the tropical hypersurfaces associated to $g_1, \dots, g_n$ intersect transversally.

Theorem 2.19 (Bihan)**.** *The number $\sharp\{V_{g_1}^{\text{trop}} \cap \cdots \cap V_{g_n}^{\text{trop}}\}$ is less or equal to the discrete mixed volume $D(\mathcal{W}_1, \dots, \mathcal{W}_n)$.*

Obviously, we have

$$\sharp\{V_{g_1,+}^{\text{trop}} \cap \cdots \cap V_{g_n,+}^{\text{trop}}\} \leq \sharp\{V_{g_1}^{\text{trop}} \cap \cdots \cap V_{g_n}^{\text{trop}}\}$$

Moreover, Theorem 1.4 of [Bih14] states that for any finite sets $\mathcal{W}_1, \dots, \mathcal{W}_r \subset \mathbb{R}^n$, we have

$$D(\mathcal{W}_1, \dots, \mathcal{W}_r) \leq \prod_{i \in [r]} (|\mathcal{W}_i| - 1).$$

Combining the latter result with Theorem 2.19 shows that Kushnirenko's conjecture is true for polynomial systems constructed by the combinatorial patchworking method of Viro, or equivalently, for tropical polynomial systems given by transversal intersections of tropical hypersurfaces.

To our knowledge, we do not know if the discrete mixed volume bound is sharp for any polynomial system with n equations in n variables satisfying that the associated tropical hypersurfaces intersect transversally. An interesting direction to start, is to look at a system (2.2.2) such that all polynomials of (2.2.2) have the same support $\mathcal{W}$. For example, when $|\mathcal{W}| = 4$, then the bound of Theorem 2.19 is 3 and is sharp, see [Bih07].

When $|\mathcal{W}| = 5$ and $n = 2$, we have $D(\mathcal{W}, \mathcal{W}) = 6$. We construct using combinatorial patchworking (Theorem 2.18) a polynomial system of two equations in two variables having a total of five distinct monomials and six non-degenerate solutions in $(\mathbb{R}_{>0})^2$. Thus proving that the bound of Theorem 2.19 is sharp when $n = 2$ and $\mathcal{W}_1 = \mathcal{W}_2 = 5$.

2.2.6 Reduced systems and non-transversal intersections

Theorem 2.18 is only adapted for the case where the tropical intersections are transverse. Therefore, we need other machinery to locate the valuations of positive solutions.

2.2.6.1 Types of non-transversal cells

In Chapter 6 of this thesis, we only work with tropical hypersurfaces in dimension two. Therefore, we classify the types of mixed cells ξ in the case where two tropical plane curves intersect non-transversally at a cell ξ. Let $\mathring{\xi}$ denote the relative interior of ξ. Note that $\xi = \mathring{\xi}$ if ξ is a point. Assume that ξ is non-transversal, we distinguish three types for such ξ.

- A cell ξ is of **type (I)** if $\dim \xi = \dim \xi_1 = \dim \xi_2 = 1$.
- A cell ξ is of **type (II)** if one of the cells ξ_1, or ξ_2 is a vertex, and the other cell is an edge.
- A cell ξ is of **type (III)** if ξ_1 and ξ_2 are vertices of the corresponding tropical curves.

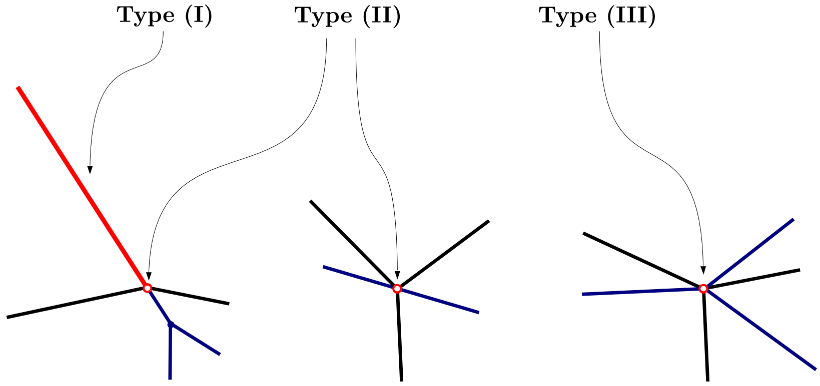

Figure 2.2: The three types of non-transversal intersection cells.

2.2.6.2 Reduced systems

Recall that for an element $a(t) \in \mathbb{K}^*$, we denote by $\operatorname{coef}(a(t))$ the non-zero coefficient corresponding to the term of $\alpha(t)$ with the smallest exponent of t.

Definition 2.20. *Let $f = \sum_{w \in \Delta(f) \cap \mathbb{Z}^2} c_w z^w$ be a polynomial in $\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]$ with $c_w \in \mathbb{K}^*$, and let ξ denote a cell of V_f^{trop}. The* ***reduced polynomial*** *$f_{|\xi} \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ of f with respect to ξ is a polynomial defined as*

$$f_{|\xi} = \sum_{w \in \Delta_\xi \cap \mathcal{W}} \operatorname{coef}(c_w) z^w,$$

where $\mathcal{W}$ is the support of f.

We extend this definition to the following. Consider a system

$$f_1(z) = f_2(z) = 0, \tag{2.2.4}$$

with f_1, f_2 in $\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]$ defined as above. Assume that the intersection set $T_1 \cap T_2$ of the tropical curves T_1 and T_2 is non-empty, and consider a mixed cell $\xi \in T_1 \cap T_2$. As explained in Subsection 2.2.4, the mixed cell ξ is written as $\xi_1 \cap \xi_2$ for some unique $\xi_1 \in T_1$ and $\xi_2 \in T_2$.

Definition 2.21. *The* ***reduced system*** *of* (4.1.1) *with respect to ξ is the system*

$$f_{1|\xi_1} = f_{2|\xi_2} = 0,$$

with $f_{i|\xi_i}$ is the reduced polynomial of f_i with respect to ξ_i for $i = 1, 2$.

In what follows, we assume that all solutions of (2.2.4) are non-degenerate. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ denote the supports of f_1 and f_2 respectively, and write

$$f_1(z) = \sum_{v \in \mathcal{W}_1} a_v z^v \quad \text{and} \quad f_2(z) = \sum_{w \in \mathcal{W}_2} b_w z^w.$$

The following result also generalizes to a polynomial system defined on the same field with n equations in n variables.

Proposition 2.22. *If the system* (2.2.4) *has a non-degenerate solution* $(\alpha,\beta)\in(\mathbb{K}^*)^2$ *such that* $\mathrm{Val}(\alpha,\beta)\in\mathring{\xi}$*, then* $(\mathrm{coef}(\alpha),\mathrm{coef}(\beta))$ *is a real solution of the reduced system*

$$f_{1|\Delta_{\xi_1}} = f_{2|\Delta_{\xi_2}} = 0. \tag{2.2.5}$$

Proof. Assume that (2.2.4) has a non-degenerate solution $(\alpha,\beta)\in(\mathbb{K}^*)^2$ such that $\mathrm{Val}(\alpha,\beta)\in\mathring{\xi}$. Since $\mathrm{Val}(\alpha,\beta)$ belongs to the relative interior of each of ξ_1 and ξ_2, we have

$$\max\{\langle \mathrm{Val}(\alpha,\beta),v\rangle+\mathrm{val}(a_v),\ v\in\mathcal{W}_1\setminus(\mathcal{W}_1\cap\Delta_{\xi_1})\}<\langle\mathrm{Val}(\alpha,\beta),v\rangle+\mathrm{val}(a_v)\quad\text{for}\quad v\in\mathcal{W}_1\cap\Delta_{\xi_1}$$

and

$$\max\{\langle \mathrm{Val}(\alpha,\beta),w\rangle+\mathrm{val}(b_w),\ w\in\mathcal{W}_2\setminus(\mathcal{W}_2\cap\Delta_{\xi_2})\}<\langle\mathrm{Val}(\alpha,\beta),w\rangle+\mathrm{val}(b_w)\quad\text{for}\quad w\in\mathcal{W}_2\cap\Delta_{\xi_2}.$$

Consequently, since $\mathrm{ord}=-\mathrm{val}$, we have $M:=-\langle\mathrm{Val}(\alpha,\beta),v\rangle-\mathrm{val}(a_v)$ and $N:=-\langle\mathrm{Val}(\alpha,\beta),w\rangle-\mathrm{val}(b_w)$ are the orders of $f_1(\alpha,\beta)$ and $f_2(\alpha,\beta)$ respectively. Therefore, replacing (z_1,z_2) by $\left(t^{\mathrm{ord}(\alpha)}z_1,t^{\mathrm{ord}(\beta)}z_2\right)$ in (2.2.4), such a system becomes

$$\begin{array}{rcl} f_1\left(t^{\mathrm{ord}(\alpha)}z_1,t^{\mathrm{ord}(\beta)}z_2\right) & = & t^M\left(\sum_{v\in\mathcal{W}_1\cap\Delta_{\xi_1}}\mathrm{coef}(a_v)z^v+g_1(z)\right),\\ f_2\left(t^{\mathrm{ord}(\alpha)}z_1,t^{\mathrm{ord}(\beta)}z_2\right) & = & t^N\left(\sum_{w\in\mathcal{W}_2\cap\Delta_{\xi_2}}\mathrm{coef}(b_w)z^w+g_2(z)\right),\end{array} \tag{2.2.6}$$

where all the coefficients of the polynomials Q_1 and Q_2 of $\mathbb{RK}[z_1^{\pm1},z_2^{\pm1}]$ have positive orders. Since (α,β) is a non-zero solution of (2.2.5), the system (2.2.6) has a non-zero solution (α_0,β_0) with $\mathrm{ord}(\alpha_0)=\mathrm{ord}(\beta_0)=0$ and $\mathrm{Coef}(\alpha,\beta)=\mathrm{Coef}(\alpha_0,\beta_0)$. It follows that taking $t>0$ small enough, we get that $\mathrm{Coef}(\alpha_0,\beta_0)$ is a non-zero solution of

$$\sum_{v\in\mathcal{W}_1\cap\Delta_{\xi_1}}\mathrm{coef}(a_v)z^v=\sum_{w\in\mathcal{W}_2\cap\Delta_{\xi_2}}\mathrm{coef}(b_w)z^w=0.$$

□

Note that Proposition 2.22 holds true for any type of tropical intersection cell ξ. However, the other direction does not always hold true when ξ is of type (I). Recall that a solution $(\alpha,\beta)\in(\mathbb{K}^*)^2$ is positive if $(\alpha,\beta)\in(\mathbb{RK}^*_{>0})^2$.

Proposition 2.23. *Assume that* $\dim\xi=0$*. If the reduced system of* (2.2.4) *with respect to* ξ *has a non-degenerate solution* $(\rho_1,\rho_2)\in(\mathbb{R}^*_{>0})^2$*, then* (2.2.4) *has a non-degenerate solution* $(\alpha,\beta)\in(\mathbb{RK}^*_{>0})^2$ *such that* $\mathrm{Val}(\alpha,\beta)=\xi$ *and* $\mathrm{Coef}(\alpha,\beta)=(\rho_1,\rho_2)$.

Proof. E. Brugallé showed in [BLdM12, Proposition 3.11] (see also [Kat09, Rab12, OP13] for more details for higher dimension and more exposition relating toric varieties and tropical intersection theory) that the number of solutions of (2.2.4) with valuation ξ is equal to the mixed volume $\mathrm{MV}(\Delta_{\xi_1},\Delta_{\xi_2})$ of ξ_1 and ξ_2 (recall that $\Delta_\xi=\Delta_{\xi_1}+\Delta_{\xi_2}$). Since we assumed that (2.2.4) has only non-degenerate solutions in $(\mathbb{K}^*)^2$, we get $\mathrm{MV}(\Delta_{\xi_1},\Delta_{\xi_2})$ distinct solutions of the system (2.2.4) in $(\mathbb{K}^*)^2$ with given valuation ξ. By Proposition 2.22, if $f_1(z)=f_2(z)=0$ and $\mathrm{Val}(z)=\xi$, then $\mathrm{Coef}(z)$ is a solution of the reduced system of (2.2.4) with respect to ξ. The number of solutions of the reduced system in $(\mathbb{C}^*)^2$ is $\mathrm{MV}(\Delta_{\xi_1},\Delta_{\xi_2})$. Assuming that this reduced system has $\mathrm{MV}(\Delta_{\xi_1},\Delta_{\xi_2})$ distinct solutions in $(\mathbb{C}^*)^2$, we obtain that the map $z\mapsto\mathrm{Coef}(z)$ induces a bijection from the set of solutions of (2.2.4) in $(\mathbb{K}^*)^2$ with valuation ξ onto the set of solutions in $(\mathbb{C}^*)^2$ of the reduced system of (2.2.4) with respect to ξ.

If z is a solution of (2.2.4) in $(\mathbb{K}^*)^2$ with $\mathrm{Val}(z) = \xi$ and $\mathrm{Coef}(z) \in (\mathbb{R}^*)^2$, then $z \in (\mathbb{R}\mathbb{K}^*)^2$ since otherwise, $z, \bar{z}$ would be two distinct solutions of (2.2.4) in $(\mathbb{K}^* \setminus \mathbb{R}\mathbb{K}^*)^2$ such that $\mathrm{Val}(z) = \mathrm{Val}(\bar{z}) = \xi$ and $\mathrm{Coef}(z) = \mathrm{Coef}(\bar{z})$. $\square$

Chapter 3

Intersecting a sparse plane curve and a line

We prove in Section 3.2 the following result.

Theorem 3.1. *Let $f \in \mathbb{R}[x,y]$ be a polynomial with at most t non-zero terms and let a, b be any real numbers. Assume that the polynomial $g(x) = f(x, ax+b)$ is not identically zero. Then g has at most $6t-7$ real roots counted with multiplicities except for the possible roots 0 and $-a/b$ that are counted at most once.*

In Section 3.3, we construct the equation (3.3.4) proving the following.

Theorem 3.2. *The maximal number of real intersection points of a real line with a real plane curve defined by a polynomial with three non-zero terms is eleven.*

3.1 Preliminary results

We present some results of M. Avendaño [Ave09] and add other ones. Consider a non-zero univariate polynomial $f(x) = \sum_{i=0}^{d} a_i x^i$ with real coefficients. Denote by $V(f)$ the number of change signs in the ordered sequence $(a_0, \ldots, a_d)$ disregarding the zero terms. Recall that the famous Descartes' rule of signs asserts that the number of (strictly) positive roots of f counted with multiplicities does not exceed $V(f)$.

Lemma 3.3. *[Ave09] We have $V((x+1)f) \leq V(f)$.*

The following result is straighforward.

Lemma 3.4. *[Ave09] If $f, g \in \mathbb{R}[x]$ and g has t terms, then $V(f+g) \leq V(f) + 2t$.*

Denote by $\mathcal{N}(h)$ the Newton polytope of a polynomial h and by $\mathring{\mathcal{N}}(h)$ the interior of $\mathcal{N}(h)$.

Lemma 3.5. *If $f, g \in \mathbb{R}[X]$, g has t terms and $V(f+g) = V(f) + 2t$, then $\mathcal{N}(g)$ is contained in $\mathring{\mathcal{N}}(f)$.*

Proof. Assume that $\mathcal{N}(g)$ is not contained in $\overset{\circ}{\mathcal{N}}(f)$. Writing $f(x) = \sum_{i=1}^{s} a_i x^{\alpha_i}$ and $g(x) = \sum_{j=1}^{t} b_j x^{\beta_j}$ with $0 \leq \alpha_1 < \cdots < \alpha_s$ and $0 \leq \beta_1 < \cdots < \beta_t$, we get $\beta_1 \leq \alpha_1$ or $\alpha_s \leq \beta_t$. Assume that $\beta_1 \leq \alpha_1$ (the case $\alpha_s \leq \beta_t$ is symmetric). Then, obviously

$$V(f(x) + g(x)) \leq 1 + V(f(x) + g(x) - b_1 x^{\beta_1}).$$

By Lemma 3.4 we have

$$V(f(x) + g(x) - b_1 x^{\beta_1}) \leq V(f) + 2(t-1).$$

All together this gives $V(f+g) \leq 1 + V(f) + 2(t-1) = V(f) + 2t - 1$. □

Proposition 3.6. *[Ave09] If $f \in \mathbb{R}[x,y]$ has t non-zero terms, then*

$$V(f(x, x+1)) \leq 2t - 2.$$

Proof. Write $f(x,y) = \sum_{k=1}^{n} a_k(x) y^{\alpha_k}$, with $0 \leq \alpha_1 < \cdots < \alpha_n$ and $a_k(x) \in \mathbb{R}[x]$. Denote by t_k the number of non-zero terms of $a_k(x)$. Define

$$f_k(x,y) = \sum_{j=k}^{n} a_j(x) y^{\alpha_j - \alpha_k}\,,\ k = 1, \ldots, n,$$

and $f_{n+1} = 0$. Then $f_k(x, x+1) = (x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1) + a_k(x)$ for $k = 1, \ldots, n-1$ and $f_n(x, x+1) = a_n(x)$. Therefore, $V(f_k(x, x+1)) \leq V(f_{k+1}(x, x+1)) + 2t_k$ by Lemma 3.3 and Lemma 3.4. Finally, $V(f(x, x+1)) \leq V(f_1(x, x+1))$ since $f(x, x+1) = (x+1)^{\alpha_1} f_1(x, x+1)$. We conclude that $V(f(x, x+1))) \leq -2 + 2(t_1 + \cdots + t_n) = 2t - 2$.

□

Proposition 3.7. *Let $f \in \mathbb{R}[x,y]$ be a polynomial with t non-zero terms. Write it as $f(x,y) = \sum_{i=1}^{t} b_i x^{\beta_i} y^{\gamma_i}$ with $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_t$. If $V(f(x, x+1)) = 2t - 2$, then*

$$\mathcal{N}(b_i x^{\beta_i} (x+1)^{\gamma_i}) \subset \overset{\circ}{\mathcal{N}}(b_t x^{\beta_t} (x+1)^{\gamma_t})$$

(in other words, $\beta_t < \beta_i \leq \beta_i + \gamma_i < \beta_t + \gamma_t$) for $i = 1, \ldots, t-1$.

Proof. We use the proof of Proposition 3.6 keeping its notations. Write $f(x,y) = \sum_{k=1}^{n} a_k(x) y^{\alpha_k}$ with $0 \leq \alpha_1 < \cdots < \alpha_n$ and assume that $V(f(x, x+1)) = 2t - 2$. It follows from the proof of Proposition 3.6 that

$$V(f_k(x, x+1)) = V(f_{k+1}(x, x+1)) + 2t_k\,, \quad k = 1, \ldots, n. \tag{3.1.1}$$

Recall that $f_k(x, x+1) = (x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1) + a_k(x)$ for $k \leq n-1$. By Lemma 3.5 and (3.1.1) we get $\mathcal{N}(a_k(x)) \subset \overset{\circ}{\mathcal{N}}((x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1))$ and thus

$$\mathcal{N}(a_k(x)(x+1)^{\alpha_k}) \subset \overset{\circ}{\mathcal{N}}((x+1)^{\alpha_{k+1}} f_{k+1}(x, x+1)) \tag{3.1.2}$$

for $k = 1, \dots, n-1$. We now show by induction on $n - k \geq 1$ that

$$\mathring{\mathcal{N}}((x+1)^{\alpha_{k+1}} f_{k+1}(x, x+1)) \subset \mathring{\mathcal{N}}(a_n(x)(x+1)^{\alpha_n}). \tag{3.1.3}$$

Together with (3.1.2) this will imply $\mathcal{N}(a_k(x)(x+1)^{\alpha_k}) \subset \mathring{\mathcal{N}}(a_n(x)(x+1)^{\alpha_n})$ for $k = 1, \dots, n-1$, and thus $\mathcal{N}(b_i x^{\beta_i}(x+1)^{\gamma_i}) \subset \mathring{\mathcal{N}}(b_t x^{\beta_t}(x+1)^{\gamma_t})$ for $i = 1, \dots, t-1$. For $n - k = 1$ the inclusion (3.1.3) is obvious. Since $f_k(x, x+1) = (x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1) + a_k(x)$ and $\mathcal{N}(a_k(x)) \subset \mathring{\mathcal{N}}((x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1))$, we get $\mathring{\mathcal{N}}(f_k(x, x+1)) = \mathring{\mathcal{N}}((x+1)^{\alpha_{k+1}-\alpha_k} f_{k+1}(x, x+1))$. Assuming (3.1.3) is true for k (hypothesis induction), this immediately gives $\mathring{\mathcal{N}}((x+1)^{\alpha_k} f_k(x, x+1)) \subseteq \mathring{\mathcal{N}}(a_n(x)(x+1)^{\alpha_n})$ and thus (3.1.3) is proved for $k - 1$. □

3.2 Proof of Theorem 3.1

We first recall the proof of the bound $6t - 4$ in [Ave09]. Let $f(x, y) = \sum_{i=1}^{t} b_i x^{\beta_i} y^{\gamma_i} \in \mathbb{R}[x, y]$ be a polynomial with at most t non-zero terms, and let a, $b \in \mathbb{R}$. Set $g(x) = f(x, ax + b)$. If $a = 0$ or $b = 0$, then f has at most t non-zero terms and Descartes' rule of signs implies that either $g = 0$ or g has at most $2t - 1 \leq 6t - 4$ real roots (counted with multiplicities except for the possible root 0). If $ab \neq 0$, then the real roots of $f(x, ax + b)$ correspond bijectively to the real roots of $f(bx/a, b(x+1)) = \hat{f}(x, x+1)$, where $\hat{f}(x, y) = \sum_{i=1}^{t} b_i a^{-\beta_i} b^{\beta_i+\gamma_i} x^{\beta_i} y^{\gamma_i}$. Since this bijection preserves multiplicities and maps the possible roots 0 and $-b/a$ of g to the roots 0 and -1 of $\hat{f}(x, x+1)$, it suffices to consider the case $a = b = 1$, i.e. $g(x) = f(x, x+1)$. So we now consider $g(x) = f(x, x+1)$. Assume that $g \neq 0$ and denote by d the degree of g.

Descartes' rule of signs and Proposition 3.6 imply that the number of positive roots of g counted with multiplicities is at most $2t - 2$. The roots of g in $]-\infty, -1[$ correspond bijectively to the positive roots of $g(-1-x) = f(-1-x, -x) = \sum_{i=1}^{t} b_i (-1)^{\beta_i+\gamma_i} x^{\gamma_i} (x+1)^{\beta_i}$. Therefore, by Proposition 3.6 the number of roots (counted with multiplicities) of g in $]-\infty, -1[$ cannot exceed $2t - 2$. Finally, the roots of g in $]-1, 0[$ correspond bijectively to the positive roots of $(x+1)^d g(\frac{-x}{x+1}) = (x+1)^d f(\frac{-x}{x+1}, \frac{1}{x+1}) = \sum_{i=1}^{t} b_i (-1)^{\beta_i} x^{\beta_i} (x+1)^{d-\beta_i-\gamma_i}$. Thus, by Proposition 3.6 there are at most $2t - 2$ such roots.

All together, this leads to the conclusion that g has at most $3(2t-2) + 2 = 6t - 4$ real roots counted with multiplicities except for the possible roots 0 and -1 that are counted at most once.

We now start the proof of Theorem 3.1.

Set $I_1 =]0, +\infty[$, $I_2 =]-\infty, -1[$ and $I_3 =]-1, 0[$. For $h \in \mathbb{R}[x]$ define

$$V_{I_1}(h) = V(h), \quad V_{I_2}(h) = V(h(-1-x)) \quad \text{and}$$

$$V_{I_3}(h) = V\left((x+1)^{\deg(h)} h\left(\frac{-x}{x+1}\right)\right).$$

By Descartes' rule of sign the number of roots of h in I_i does not exceed $V_{I_i}(h)$. To prove Theorem 3.1, it suffices to show that

$$V_{I_1}(g) + V_{I_2}(g) + V_{I_3}(g) \leq 3(2t-2) - 3 \tag{3.2.1}$$

Define polynomials

$$h_1(x) = x^d h\left(\frac{1}{x}\right), \quad h_2(x) = (x+1)^d h\left(\frac{-x}{x+1}\right) \quad \text{and} \quad h_3(x) = h(-1-x)$$

so that $V_{I_1}(h_1) = V_{I_1}(h)$, $V_{I_1}(h_2) = V_{I_3}(h)$ and $V_{I_1}(h_3) = V_{I_2}(h)$.

Lemma 3.8. *For any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, we have*

$$V_{I_i}(h_i) = V_{I_i}(h) \quad \textit{and} \quad V_{I_i}(h_j) = V_{I_k}(h)$$

Proof. We have $h_1(-x-1) = (-1)^d(x+1)^d h\left(-\frac{1}{x+1}\right)$. Therefore
$V(h_1(-x-1)) = V\left((x^{-1}+1)^d h\left(-\frac{1}{x^{-1}+1}\right)\right)$, thus

$$V(h_1(-x-1)) = V\left(\left(\frac{x+1}{x}\right)^d h\left(-\frac{x}{x+1}\right)\right) = V\left((x+1)^d h\left(-\frac{x}{x+1}\right)\right),$$

and we get $V_{I_2}(h_1) = V_{I_3}(h)$. We have $(x+1)^d h_1\left(-\frac{x}{x+1}\right) = (-x)^d h(-1-x^{-1})$ from which we obtain $V_{I_3}(h_1) = V_{I_2}(h)$.

Equalities $V_{I_2}(h_2) = V_{I_2}(h)$ and $V_{I_3}(h_2) = V_{I_1}(h)$ follow from
$h_2(-1-x) = (-x)^d h(-1-x^{-1})$ and $(x+1)^d h_2(-\frac{x}{x+1}) = h(x)$.

Finally, $V_{I_2}(h_3) = V_{I_1}(h)$ comes from $h_3(-x-1) = h(x)$ and $V_{I_3}(h_3) = V_{I_3}(h)$ is a consequence of $(x+1)^d h_3(-\frac{x}{x+1}) = (x+1)^d h(-\frac{1}{x+1})$ and the equality $V((x+1)^d h(-\frac{1}{x+1})) = V_{I_3}(h)$ shown above. □

We now proceed to the proof of (3.2.1). We already know that $V_{I_i}(g) \leq 2t-2$ for $i = 1, 2, 3$. If $V_{I_i}(g) \leq 2t-3$ for all i, then (3.2.1) is trivially true. With the help of Lemma 3.8, it suffices now to show that if $V_{I_1}(g) = 2t-2$ then $V_{I_2}(g) \leq 2t-3$, $V_{I_3}(g) \leq 2t-3$, and $V_{I_2}(g)+V_{I_3}(g) < 2(2t-3)$. So assume $V_{I_1}(g) = 2t-2$. Then by Proposition 3.7

$$\beta_t < \beta_i \leq \beta_i + \gamma_i < \beta_t + \gamma_t, \ , \quad i = 1, \ldots, t-1. \tag{3.2.2}$$

We have $g(-1-x) = \sum_{i=1}^t b_i(-1)^{\beta_i+\gamma_i} x^{\gamma_i}(x+1)^{\beta_i}$. Recall that $V_{I_2}(g) = V(g(-x-1)) \leq 2t-2$ by Proposition 3.6. From (3.2.2), we get $\gamma_t > \gamma_i$ for $i = 1, \ldots, t-1$. It follows then from Proposition 3.7 that $V(g(-x-1)) \leq 2t-3$.

Write $g(-1-x) = \tilde{g}(-x-1) + b_t(-1)^{\beta_t+\gamma_t} x^{\gamma_t}(x+1)^{\beta_t}$, and then $g(-1-x)(x+1)^{-\beta_t} = \tilde{g}(-x-1)(x+1)^{-\beta_t} + b_t(-1)^{\beta_t+\gamma_t} x^{\gamma_t}$. We note that (3.2.2) implies $\beta_t < \beta_i$ for $i = 1, \ldots, t-1$, so that both members of the previous equality are polynomials. Moreover, from (3.2.2) we also get $\beta_i - \beta_t + \gamma_i < \gamma_t$, and thus γ_t does not belong to the Newton polytope of the polynomial $\tilde{g}(-x-1)(x+1)^{-\beta_t}$. It follows that $V(g(-1-x)(x+1)^{-\beta_t}) \leq V(\tilde{g}(-x-1)(x+1)^{-\beta_t}) + 1$. By Lemma 3.3 we have $V(g(-1-x)) \leq V(g(-x-1)(x+1)^{-\beta_t})$. Therefore, $V(g(-1-x)) \leq V(\tilde{g}(-x-1)(x+1)^{-\beta_t}) + 1$. On the other hand Proposition 3.6 yields $V(\tilde{g}(-x-1)(x+1)^{-\beta_t}) \leq 2(t-1) - 2 = 2t-4$.

Therefore, if $V(g(-1-x)) = 2t-3$, then $V(\tilde{g}(-x-1)(x+1)^{-\beta_t}) = 2t-4$, and we may apply Proposition 3.7 to $\tilde{g}(-x-1)(x+1)^{-\beta_t}$ in order to get

$$\gamma_{i_0} < \gamma_i \leq \gamma_i + \beta_i < \gamma_{i_0} + \beta_{i_0} \text{ for all } i = 1, \ldots, t-1 \text{ and } i \neq i_0, \tag{3.2.3}$$

where i_0 is determined by $\beta_{i_0} \geq \beta_i$ for $i = 1, \ldots, t-1$.

Starting with $g_1(x) = x^d g(1/x) = \sum_{i=1}^t b_i x^{d-\beta_i-\gamma_i}(x+1)^{\gamma_i}$ instead of g in the previous computation, we obtain that if $V(g_1) = 2t-2$ then $V_{I_2}(g_1) \leq 2t-3$ and if $V_{I_2}(g_1) = 2t-3$, then the substitution of $d - \beta_i - \gamma_i$ for β_i in (3.2.3) holds true:

$$\gamma_{i_1} < \gamma_i \leq d - \beta_i < d - \beta_{i_1} \text{ for all } i = 1, \ldots, t-1 \text{ and } i \neq i_1, \tag{3.2.4}$$

where i_1 is determined by $d-\beta_{i_1}-\gamma_{i_1} \geq d-\beta_i-\gamma_i$ for $i=1,\ldots,t-1$.

On the other hand, $V(g)=V(g_1)$ and $V(g_1(-x-1))=V_{I_2}(g_1)=V_{I_3}(g)$ by Lemma 3.8. Thus if $V(g)=2t-2$ then $V_{I_3}(g)\leq 2t-3$ and if $V_{I_3}(g)=2t-3$, then formula (3.2.4) holds true. It turns out that (3.2.3) and (3.2.4) are incompatible. Indeed, if (3.2.3) and (3.2.4) hold true simultaneously, then $i_0=i_1$ but then (3.2.4) implies that $\gamma_{i_0}+\beta_{i_0}<\gamma_i+\beta_i$ for all $1,\ldots,t-1$ with $i\neq i_0$ which contradicts (3.2.3). Consequently, if $V(g)=V_{I_1}(g)=2t-2$, then $V_{I_2}(g)\leq 2t-3$, $V_{I_3}(g)\leq 2t-3$ and $V_{I_2}(g)+V_{I_3}(g)<2(2t-3)$.

3.3 Optimality

We prove that the bound in Theorem 3.1 is sharp for $t=3$ (Theorem 3.2). We look for a polynomial $P\in\mathbb{R}[x,y]$ with three non-zero terms such that $P(x,x+1)$ has nine real roots distinct from 0 and -1. It follows from the previous section that if such P exists then, either $P(x,x+1)$ has three roots in each interval I_1, I_2 and I_3, or $P(x,x+1)$ has four roots in one interval, three roots in another interval, and two roots in the last one. We give necessary conditions for the second case, which thanks to Lemma 3.8 reduces to the case where $P(x,x+1)$ has four roots in $I_1=]0,+\infty[$, three roots in $I_3=]-1,0[$ and two roots in $I_2=]-\infty,-1[$.

Multiplication of P by a monomial does not alter the roots of $P(x,x+1)$ in $\mathbb{R}\setminus\{0,-1\}$, so dividing by the smallest power of x, we may assume that P has the following form

$$P(x,y)=ay^{l_1}+bx^{k_2}y^{l_2}+x^{k_3}y^{l_3},$$

where k_2, k_3, l_1, l_2, l_3 are nonnegative integer numbers and a,b are real numbers.

Lemma 3.9. *If $P(x,x+1)$ has four real positive roots, then $k_2>0$, $k_3>0$, $l_1>l_2+k_2$ and $l_1>l_3+k_3$.*

Proof. If $P(x,x+1)$ has four real positive roots, then $V(P(x,x+1))=4$. Rewriting $P(x,x+1)=\sum_{i=1}^{3}b_ix^{\beta_i}(x+1)^{\gamma_i}$ with $0\leq\gamma_1\leq\gamma_2\leq\gamma_3$, Proposition 3.7 yields $\beta_3<\beta_i\leq\beta_i+\gamma_i<\beta_3+\gamma_3$ for $i=1,2$. Since k_2 and k_3 are nonnegative, we get $\beta_3=0$, $k_2,k_3>0$ and $\beta_3+\gamma_3=\gamma_3=l_1$, so $l_1>\max(l_2+k_2,l_3+k_3)$. □

Since $l_1>l_2$ and $l_1>l_3$, we may divide $P(x,x+1)$ by $(x+1)^{l_2}$ or $(x+1)^{l_3}$ to get a polynomial equation with the same solutions in $\mathbb{R}\setminus\{0,-1\}$. So without loss of generality we may assume that

$$P(x,x+1)=a(x+1)^{l_1}+bx^{k_2}(x+1)^{l_2}+x^{k_3}, \tag{3.3.1}$$

where $k_2,k_3>0$, $l_2\geq 0$, $l_1>k_2+l_2$ and $l_1>k_3$.

Lemma 3.10. *Assume that the polynomial* (3.3.1) *has four roots in I_1, and three roots in I_3 or I_2. Then k_3 does not belong to the interval $[k_2,k_2+l_2]$. Moreover, we have $a<0$ and $b>0$.*

Proof. We prove that if $k_2\leq k_3\leq k_2+l_2$, then (3.3.1) has at most two roots in I_2 and in I_3.

The roots in I_2 are in bijection with the positive roots of

$$P(-x-1,-x)=(-1)^{l_1}ax^{l_1}+(-1)^{k_2+l_2}bx^{l_2}(x+1)^{k_2}+(-1)^{k_3}(1+x)^{k_3}.$$

Recall that $l_2\geq 0$. If $k_2\leq k_3\leq k_2+l_2$ then Proposition 3.7 yields $V((-1)^{k_2+l_2}bx^{l_2}(x+1)^{k_2}+(-1)^{k_3}(1+x)^{k_3})\leq 1$. Now, since $l_1>k_2+l_2$ and $l_1>k_3$, we get $V(P(-x-1,-x))\leq 2$, and thus (3.3.1) has at most two roots in I_2.

The roots in I_3 are in bijection with the positive roots of

$$(1+x)^{l_1}P(\frac{-x}{x+1},\frac{-x}{x+1}+1)=a+b(-1)^{k_2}x^{k_2}(1+x)^{l_1-k_2-l_2}+(-1)^{k_3}x^{k_3}(1+x)^{l_1-k_3}$$

From $k_3 \leq k_2+l_2$, we get $l_1-k_2-l_2 \leq l_1-k_3$. Thus, Proposition 3.7 together with $k_2 \leq k_3$ yields $V(b(-1)^{k_2}x^{k_2}(1+x)^{l_1-k_2-l_2}+(-1)^{k_3}x^{k_3}(1+x)^{l_1-k_3}) \leq 1$. From $k_2,k_3>0$ we get $V((1+x)^{l_1}P(\frac{-x}{x+1},\frac{-x}{x+1}+1)) \leq 2$, and thus (3.3.1) has at most two roots in I_3.

Finally, if (3.3.1) has four positive roots, then obviously $ab<0$. If k_3 does not belong to $[k_2,k_2+l_2]$ and $a>0$, then $V((x+1)^{l_1}+bx^{k_2}(x+1)^{l_2}+x^{k_3}) = V((x+1)^{l_1}+bx^{k_2}(x+1)^{l_2})$ (recall that $k_2 \leq k_2+l_2<l_1$). But the second sign variation is a most two by Proposition 3.6. We conclude that $a<0$ and $b>0$. □

Lemma 3.11. *Assume that the polynomial* (3.3.1) *has four roots in* I_1*, two roots in* I_2 *and three roots in* I_3*. Assume furthermore that* $k_3<k_2$*. Then,* l_1 *is odd,* k_2 *is odd,* k_3 *is even and* l_2 *is even.*

Proof. Since (3.3.1) has exactly nine real roots counted with multiplicity, its degree l_1 is odd. We have already seen that if (3.3.1) has four roots in $I_1=]0,+\infty[$, two roots in $I_2=]-\infty,-1[$ and three roots in $I_3=]-1,0[$, then $a<0$, $b>0$, $l_1>l_2$ and $k_3 \notin [k_2,k_2+l_2]$. Assume from now on that $k_3<k_2$.

Since (3.3.1) has two roots in $I_2=]-\infty,-1[$, we have $V(P(-x-1,-x)) \geq 2$, where $P(-x-1,-x)=(-1)^{k_3}(1+x)^{k_3}+(-1)^{k_2+l_2}bx^{l_2}(x+1)^{k_2}+(-1)^{l_1}ax^{l_1}$. But since $k_3<k_2 \leq k_2+l_2<l_1$, we get that $(-1)^{k_3}\cdot(-1)^{k_2+l_2}b<0$ and $(-1)^{k_2+l_2}b\cdot(-1)^{l_1}a<0$. Using $a<0$ and $b>0$, we obtain that k_2+l_2 is odd and k_3 is even.

Since (3.3.1) has three roots in $I_3=]-1,0[$, we have $V((1+x)^{l_1}P(\frac{-x}{x+1},\frac{-x}{x+1}+1)) \geq 3$, where $(1+x)^{l_1}P(\frac{-x}{x+1},\frac{-x}{x+1}+1)=a+b(-1)^{k_2}x^{k_2}(1+x)^{l_1-k_2-l_2}+(-1)^{k_3}x^{k_3}(1+x)^{l_1-k_3-l_3}$. We know that k_3 is even and that $b>0$. Thus in order to get coefficients with different signs in $b(-1)^{k_2}x^{k_2}(1+x)^{l_1-k_2-l_2}+(-1)^{k_3}x^{k_3}(1+x)^{l_1-k_3-l_3}$, the integer k_2 should be odd. Since we know that k_2+l_2 is odd, this gives that l_2 is even. □

Assume now that (3.3.1) has four roots in I_1, two roots in I_2 and three roots in I_3. Then $a<0$, $b>0$ and k_3 does not belong to $[k_2,k_2+l_2]$ by Lemma 3.10. Assume that $k_3<k_2$. Then l_1 is odd, k_2 is odd, k_3 is even and l_2 is even by Lemma 3.11. The roots of (3.3.1) are solutions to the equation $f(x)=-a$, where $f(x)=bx^{k_2}(1+x)^{l_2-l_1}+x^{k_3}(1+x)^{-l_1}$. Since the rational function f has no pole outside $\{-1,0\}$, by Rolle's Theorem its derivative has at least three roots in I_1, one root in I_2 and two roots in I_3. We compute that $f'(x)=0$ is equivalent to $\Phi(x)=1$, where Φ is the rational map

$$\Phi(x)=\frac{-bx^{k_2-k_3}(1+x)^{l_2}A_1(x)}{A_2(x)}, \tag{3.3.2}$$

with $A_1(x)=(k_2+l_2-l_1)x+k_2$ and $A_2(x)=(k_3-l_1)x+k_3$. From $0<k_3<k_2$, $l_2 \geq 0$ and $l_1>0$, we obtain that the roots of A_1 and A_2 satisfy $0<\frac{k_3}{l_1-k_3}<\frac{k_2}{l_1-k_2-l_2}$. Moreover, the roots of Φ are -1 with even multiplicity l_2, 0 with odd multiplicity k_2-k_3 and the positive root of A_1 (which is a simple root of Φ). The poles of Φ are the positive root of A_2 and the point at infinity which has multiplicity $\deg(\Phi)-1$ if we homogeinize Φ into a rational map from the Riemann sphere $\mathbb{C}P^1$ to itself.

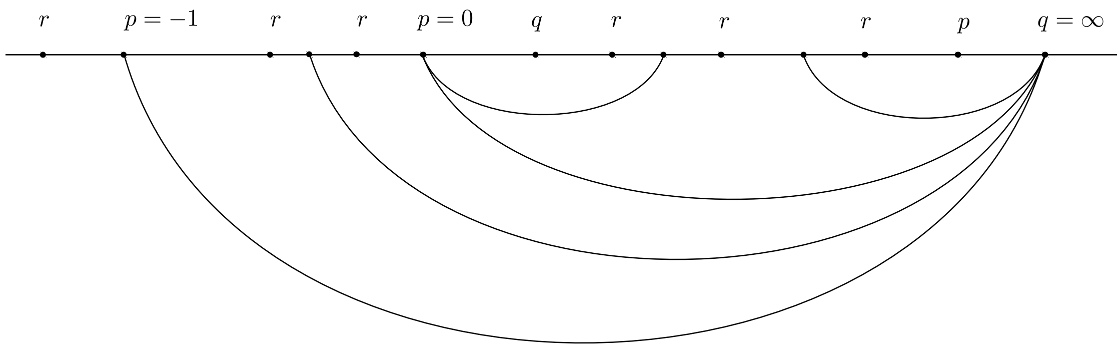

Figure 3.1: A real dessin d'enfant for φ.

We find exact values of coefficients and exponents of (3.3.2) in the following way. Note that the exponents of (3.3.2) are independent of l_1. We first choose small values $k_2 = 5$, $k_3 = 2$, $l_2 = 2$ satisfying the above parity conditions. Then, we look for a function

$$\varphi(x) = \frac{cx^3(x+1)^2(x-\rho_1)}{x-\rho_2}, \tag{3.3.3}$$

such that c is some real constant, $0 < \rho_2 < \rho_1$ and $\varphi(x) = 1$ has three solutions in I_1, one solution in I_2 and two solutions in I_3.

The existence of such a function φ is certified by Figure 3.1 thanks to Proposition 2.2. Figure 3.1 shows $H\Gamma$ contained in one connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. From Figure 3.1, we see that $0 < \rho_2 < \rho_1$ and that φ has the desired number of inverse images (letters r) of 1 in each interval I_i.

Now we want to identify (3.3.3) and (3.3.2). Recall that $k_2 = 5$, $k_3 = 2$, $l_2 = 2$ are fixed. We look at the function $\frac{x^3(x+1)^2(x-\rho_1)}{x-\rho_2}$, where $\rho_1 = \frac{k_2}{l_1-k_2-l_2}$ and $\rho_2 = \frac{k_3}{l_1-k_3}$, and increase l_1 so that some level set of this function has three solutions in I_1, one solution in I_2 and two solutions in I_3. It turns out that $l_1 = 17$ is large enough and the level set gives the value 29 for b. Finally, integrating Φ and choosing $a = -0,002404$, we get

$$-0.002404(x+1)^{17} + 29x^5(x+1)^2 + x^2 \tag{3.3.4}$$

for (3.3.1). This polynomial has four roots in I_1, two roots in I_2 and three roots in I_3. This has been computed using SAGE version 6.6 which gives the following approximated roots: 0.18859, 0.22206, 0.25196, 0.44416 in I_1, -3.96032, -1.15048 in I_2, and -0.61459, -0.58528,-0.03594 in I_3.

Multiplying this polynomial by $x(x+1)$ gives a polynomial of the form $P(x, x+1)$ (where $P \in \mathbb{R}[x,y]$ has three non-zero terms) having eleven real roots.

Chapter 4

Positive intersection points of a trinomial and a t-nomial curves

4.1 Introduction and statement of the main results

Consider a system

$$f = g = 0, \tag{4.1.1}$$

where f has $t \geq 3$ non-zero terms and g has three non-zero terms. We assume in this chapter that (4.1.1) has a finite number of solutions, and denote by $\mathcal{S}(3,t)$ the maximal number of non-degenerate positive solutions such a system can have. We prove the following result in Section 4.2.

Theorem 4.1. *We have* $\mathcal{S}(3,t) \leq 3 \cdot 2^{t-2} - 1$.

Consider now a function

$$\phi(x) = \frac{x^{\alpha}(1-x)^{\beta}P(x)}{Q(x)},$$

where $\alpha, \beta \in \mathbb{Q}$, and both P and Q are real polynomials. Using real dessins d'enfant, we prove in Section 4.3 the following result.

Theorem 4.2. *We have* $\sharp\{x \in]0,1[\ |\phi(x) = 1\} \ \leq \deg P + \deg Q + 2$.

We say that two triangles Δ_1 and Δ_2 in $\mathbb{R}^2$ alternate when any two consecutive edges of their Minkowski sum $\Delta_1 + \Delta_2$ are not translate of two consecutive edges of Δ_1 or of Δ_2 (see Definition 4.30). We prove in Section 4.4 the following result.

Theorem 4.3. *If a system of two trinomials in two variables has 5 positive solutions, then the Newton triangles of the respective equations do not alternate.*

4.2 Proof of Theorem 4.1

Define the polynomials f and g of (4.1.1) as

$$f(u,v)=\sum_{i=1}^{t}a_i u^{\alpha_i}v^{\beta_i} \quad \text{and} \quad g(u,v)=\sum_{j=1}^{3}b_j u^{\gamma_j}v^{\delta_j}, \tag{4.2.1}$$

where all a_i and b_i are real.

We suppose that the system (4.1.1) has positive solutions, thus the coefficients of g have different signs. Therefore without loss of generality, let $b_1=-1$, $b_2>0$ and $b_3>0$. Since we are looking for positive solutions of (4.1.1) with non-zero coordinates, one can assume that $\gamma_1=\delta_1=0$. Furthermore, the monomial change of coordinates $(u,v)\to(x,y)$ of $(\mathbb{C}^*)^2$ defined by $b_2u^{\gamma_2}v^{\delta_2}=x$ and $b_3u^{\gamma_3}v^{\delta_3}=y$ preserves the number of positive solutions. Therefore, we are reduced to a system

$$\sum_{i=1}^{t}c_i x^{k_i}y^{l_i}=-1+x+y=0, \tag{4.2.2}$$

where c_i is real for $i=1,\cdots,t$, and all k_i and l_i are rational numbers.

We now look for the positive solutions of (4.2.2). It is clear that since both x and y are positive, then $x\in]0,1[$. Substituting $1-x$ for y in (4.2.2), we get

$$F(x):=\sum_{i=1}^{t}c_i x^{k_i}(1-x)^{l_i}, \tag{4.2.3}$$

so that the number of positive solutions of (4.1.1) is equal to that of roots of F in $]0,1[$. For any $d\in\mathbb{N}$, denote by $\mathbb{R}_d[x]$ the set of real polynomials of degree at most d.

Lemma 4.4. *Consider a function defined by* $h(x)=\sum_{i=1}^{s}b_i x^{m_i}(1-x)^{n_i}h_{i,d}(x)$, *where* $h_{1,d},\ldots,h_{s,d}\in\mathbb{R}_d[x]$. *Then for all* $r\in\mathbb{N}$, *there exist* $h_{1,d+r},\ldots,h_{s,d+r}\in\mathbb{R}_{d+r}[x]$ *such that the* r*-th derivative of* h *is defined by*

$$h^{(r)}(x)=\sum_{i=1}^{s}x^{m_i-r}(1-x)^{n_i-r}h_{i,d+r}(x).$$

Proof. One computes that

$$(x^m(1-x)^n h(x))'=x^{m-1}(1-x)^{n-1}\cdot\left[\left((n-m)x+m\right)h(x)+x(1-x)h'(x)\right].$$

□

Define $f_1,\ldots,f_t$ inductively by $f_1(x)=x^{-k_1}(1-x)^{-l_1}F(x)$ and

$$f_{j+1}(x)=x^{k_j-k_{j+1}+2^{j-1}}\cdot(1-x)^{l_j-l_{j+1}+2^{j-1}}\cdot f_j^{(2^{j-1})}(x),\ j=1,\ldots,t-1.$$

Lemma 4.5. *For* $j=1,\ldots,t$, *there exist polynomials* $h_{j,d_j},\ldots,h_{t,d_j}\in\mathbb{R}_{d_j}[x]$ *such that* $d_j=2^{j-1}-1$,

$$f_j(x)=h_{j,d_j}(x)+\sum_{i=j+1}^{t}x^{k_i-k_j}(1-x)^{l_i-l_j}h_{i,d_j} \quad \textit{for} \quad j=1,\ldots,t-1 \tag{4.2.4}$$

and $f_t=h_{t,d_t}(x)$.

Proof. This follows easily from Lemma 4.4. □

Let N_j denote the set $\sharp\{x \in]0,1[\mid f_j(x) = 0\}$ for $j = 1, \dots, t$. Note that $N_1 = \sharp\{x \in]0,1[\mid F(x) = 0\}$. Rolle's Theorem implies directly that

$$N_j \leq N_{j+1} + 2^{j-1} \quad \text{for} \quad j = 1, \dots, t-1. \tag{4.2.5}$$

Moreover, $N_t \leq d_t = 2^{t-1} - 1$ by Lemma 4.5. Consequently, we get

$$\sharp\{x \in]0,1[\mid F(x) = 0\} = N_1 \leq \sum_{j=1}^{t-2} 2^{j-1} + N_{t-1} = 2^{t-2} - 1 + N_{t-1}. \tag{4.2.6}$$

By (4.2.5), we have $N_{t-1} \leq N_t + 2^{t-2} \leq 2^{t-1} - 1 + 2^{t-2}$ (since $N_t \leq 2^{t-1} - 1$), which together with (4.2.6) gives

$$\sharp\{x \in]0,1[\mid F(x) = 0\} \leq 2^t - 2.$$

This is the bound obtained in [LRW03]. The sharper bound that we give is obtained by improving the bound on N_{t-1}. This improvement uses the fact that f_{t-1} is a rational function, thus one can use a different approach to get a sharp bound on N_{t-1}. We have already seen that

$$f_{t-1}(x) = -Q(x) + x^{k_t - k_{t-1}}(1-x)^{l_t - l_{t-1}} P(x),$$

where $P, Q \in \mathbb{R}_{d_{t-1}}[x]$ with $d_{t-1} = 2^{t-2} - 1$. We have

$$f_{t-1}(x) = 0 \quad \Longleftrightarrow \quad \frac{x^{k_t - k_{t-1}}(1-x)^{l_t - l_{t-1}} P(x)}{Q(x)} = 1.$$

Therefore applying Theorem 4.2, we get $N_{t-1} \leq 2^{t-1} - 2 + 2 = 2^{t-1}$. Finally, by (4.2.5), we get

$$\sharp\{x \in]0,1[\mid f(x) = 0\} \leq 2^{t-1} + 2^{t-2} - 1 = 3 \cdot 2^{t-2} - 1,$$

which finishes the proof of Theorem 4.1 assuming Theorem 4.2.

4.3 Proof of Theorem 4.2

Consider the function

$$\phi(x) = \frac{x^\alpha (1-x)^\beta P(x)}{Q(x)},$$

where $\alpha, \beta \in \mathbb{Q}$ and $P, Q \in \mathbb{R}[x]$. Let m be a positive integer such that $m\alpha$ and $m\beta$ are integers. Then $\varphi := \phi^m$ is a rational function from $\mathbb{C}$ to $\mathbb{C}$. Here and in the rest of this chapter, we see the source and target spaces of φ as the affine charts of $\mathbb{C}P^1$ given by the non-vanishing of the first coordinate of homogeneous coordinates and denote with the same symbol φ the rational function from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ obtained by homogenization with respect to these coordinates. In what follows, we apply the theory of Groethendieck's dessin d'enfant to the rational function φ.

Denote by $\Gamma := \varphi^{-1}(\mathbb{R}P^1)$. Since the graph is invariant under complex conjugation, it is determined by its intersection with one connected component H (for half) of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. In most figures we will only show one half part $H \cap \Gamma$ together with $\mathbb{R}P^1 = \partial H$ represented as a horizontal line. Moreover, for simplicity, we omit the arrows. The reader may refer to Chapter 2 for more details on real dessins d'enfant.

Definition 4.6. *Any root or pole of φ is called a* ***special point*** *(of φ), and any other point of Γ is called* ***non-special****.*

4.3.1 Reduction to a simpler case

We first need a definition.

Definition 4.7. *Let a, b be two critical points of φ i.e. vertices of Γ. We say that a and b are **neighbours** if there is a branch of $\Gamma \setminus \mathbb{R}P^1$ joining them such that this branch does not contain any special or critical points of φ other than a or b.*

In this section, we show how to reduce to the case where φ satisfies the following properties

(i) All roots of P and Q are special points of φ with the same multiplicity m.

(ii) Each non-special critical point of φ has multiplicity two and is not a solution of $\varphi = 1$. (4.3.1)

(iii) All *real* non-special critical points of φ are neighbours to *real* critical points of φ.

We will introduce an algorithm that transforms any dessin d'enfant Γ of φ to a dessin d'enfant Γ' of a function satisfying the three properties mentioned above. Moreover, this transformation does not reduce the number of *real* letters r of φ. Therefore, to prove Theorem 4.2, it suffices to consider a function φ satisfying (4.3.1).

This algorithm is a series of transformations which are devided into two types. The first type, called type a), reduces the valencies of all critical points so they verify the conditions **(i)** and **(ii)**. The second type, called type b), transforms a couple of conjugate points p (resp. q, r, non-special critical points) into a point p (resp. q, r, non-special critical point) which belongs to $\mathbb{R}P^1$.

4.3.1.1 Transformation of type a)

Consider a critical point α of φ, which does not belong to $\{0, 1, \infty\}$.

• Assume that $\alpha \in \mathbb{R}P^1$. Let $\mathcal{U}_\alpha$ be a small neighborhood of α in $\mathbb{C}P^1$ such that $\mathcal{U}_\alpha \setminus \{\alpha\}$ does not contain letters r, critical points or special points.

Assume that α is a special point (a root or a pole of φ). Then the valency of α is equal to $2km$ for some natural number k. We transform the graph Γ inside $\mathcal{U}_\alpha$ as in Figure 4.1. In the new graph Γ', the neighborhood $\mathcal{U}_\alpha$ contains two real special points and a real non-special critical point of φ (and no other letters p, q, r and vertices). If α is a root (resp. pole) of φ then both special points are roots (resp. poles) of φ with multiplicities m and $(k-1)m$. Moreover, the new non-special critical point has multiplicity 2. It is obvious that the resulting graph Γ' is still a real dessin d'enfant.

Assume that α is a non-special critical point that is a letter r (a root of $\varphi - 1$). Then the valency of α is equal to $2k$ for some natural number $k \geq 2$. We transform the graph Γ as in Figure 4.2. In the new graph Γ', the neighborhood $\mathcal{U}_\alpha$ contains two letters r of multiplicity $2(k-1)$ and 1 respectively, and one non-special critical point of multiplicity 2, which is not a letter r (and no other letters p, q, r or vertices).

Assume that α is a non-special critical point that is *not* a letter r. Then the valency of α is equal to $2k$ for some natural number $k \geq 3$. We transform the graph Γ such that in the new graph Γ', the neighborhood $\mathcal{U}_\alpha$ contains two non-special critical points, which are not letters r, with multiplicities 2 and $(k-1)$ (and no other letters p, q, r or vertices).

• Assume now that $\alpha \notin \mathbb{R}P^1$. Consider a small neighborhood $\mathcal{U}_\alpha$ of α and the corresponding neighborhood of its conjugate $\bar{\alpha}$ (the image of $\mathcal{U}_\alpha$ by the complex conjugation). Assume that both neighborhoods are disjoint and both $\mathcal{U}_\alpha \setminus \{\alpha\}$ and $\mathcal{U}_{\bar{\alpha}} \setminus \{\bar{\alpha}\}$ do not contain letters r, critical points or special points. Recall that the valency of α is even. Choose two branches of $\Gamma \cap \mathcal{U}_\alpha$ starting from α such that the complement of these two branches in $\mathcal{U}_\alpha$ has two connected components containing the same number of branches of $\Gamma \cap \mathcal{U}_\alpha$. We transform $\Gamma \cap \mathcal{U}_\alpha$ similarly as in the case $\alpha \in \mathbb{R}P^1$ and do the corresponding transformation of the image of $\Gamma \cap \mathcal{U}_\alpha$ by the complex conjugation.

Assume that α is a special point (a root or a pole of φ). We transform the graph Γ inside $\mathcal{U}_\alpha$ as in Figure 4.3. In $\mathcal{U}_\alpha$, the resulting graph Γ' contains two special points of φ with multiplicities m and $(k-1)m$ respectively, and one non-special critical point with multiplicity 2 (and no other letters p, q, r or vertices), all of which belong to the previously chosen two branches.

Assume that α is a non-special critical point that is a letter r (a root of $\varphi - 1$). Then the valency of α is equal to $2k$ for some natural number $k \geq 2$. In the new graph Γ', the neighborhood $\mathcal{U}_\alpha$ contains two letters r of multiplicity $2(k-1)$ and 1 respectively, and one non-special critical point of multiplicity 2, which is not a letter r (and no other letters p, q, r or vertices), all of which belong to the previously chosen branches.

Assume that α is a non-special critical point that is *not* a letter r. Then the valency of α is equal to $2k$ for some natural number $k \geq 3$. We transform the graph Γ such that in the new graph Γ', the neighborhood $\mathcal{U}_\alpha$ contains two non-special critical points, which are not letters r and which belong to the previously chosen two branches, with multiplicities 2 and $(k-1)$ respectively (and no other letters p, q, r or vertices).

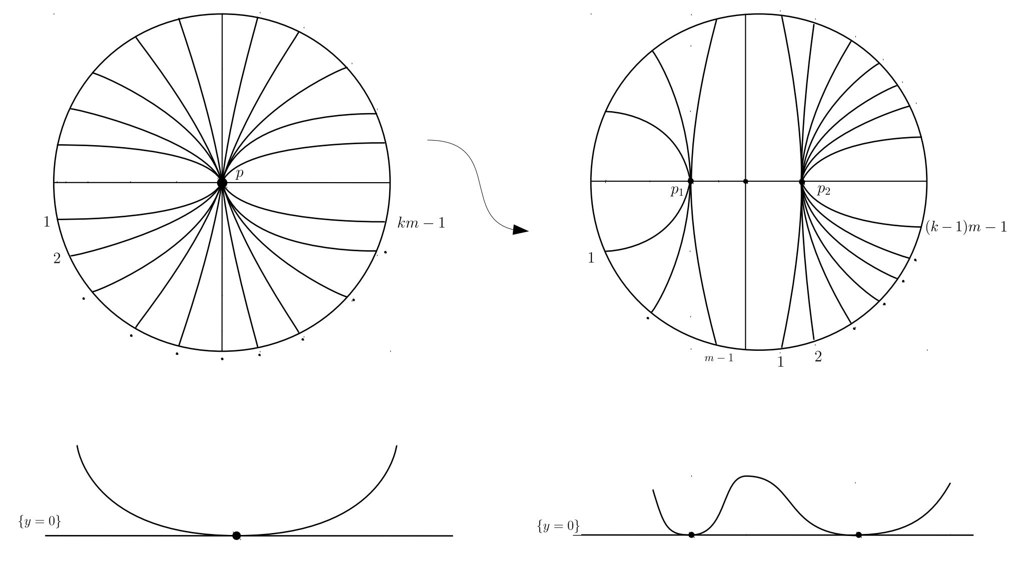

Figure 4.1: A transformation of type a) where α is a real root of P, $k = 3$ and $m = 4$.

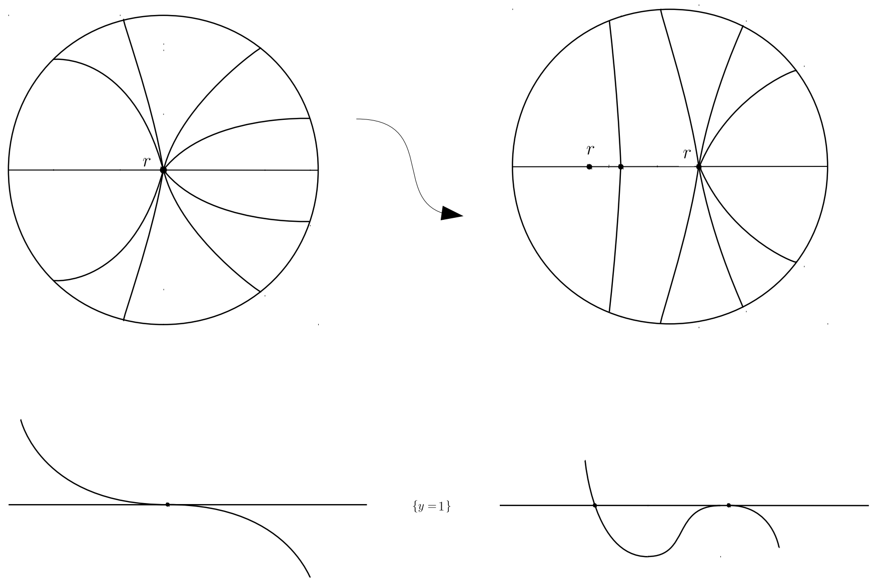

Figure 4.2: A transformation of type a) where α is a real root of $\varphi - 1$ with multiplicity 5.

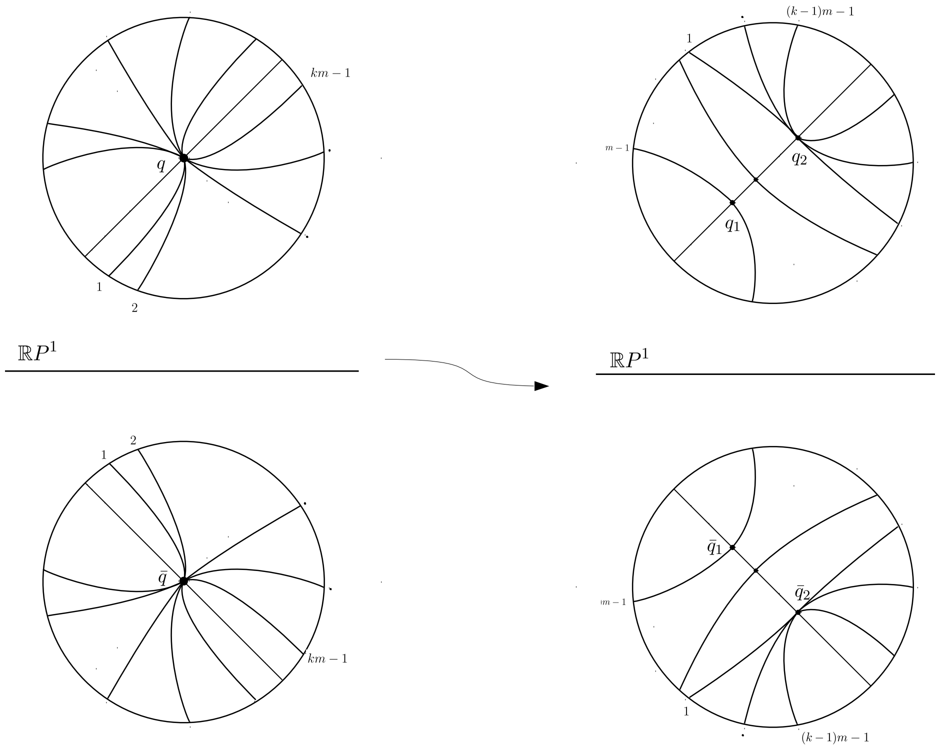

Figure 4.3: A transformation of type a) where α is a complex root of Q, $k = 3$ and $m = 2$.

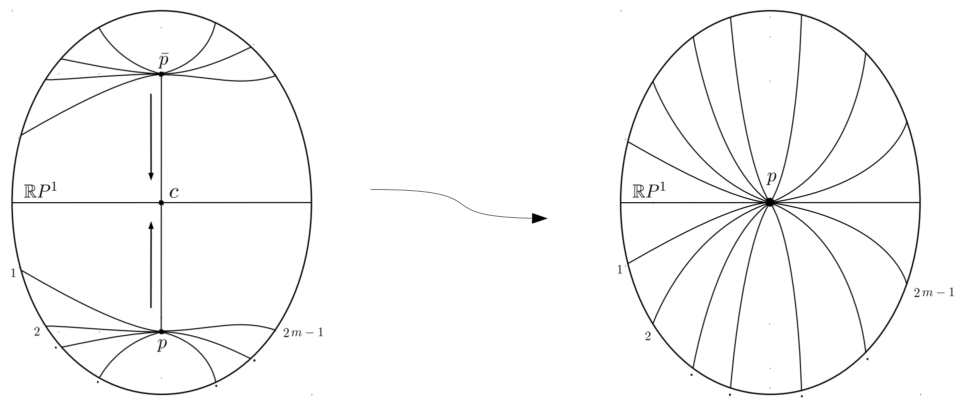

Figure 4.4: A transformation of type b) where α is a letter p, and $m = 4$.

We make this type of transformation to every point α mentioned before. Repeating this process several times gives eventually the conditions **(i)** and **(ii)**.

4.3.1.2 Transformation of type b)

Consider a point $\alpha \in \Gamma \backslash \mathbb{R}P^1$, which is either a letter p, q, r or a non-special critical point, together with its conjugate $\bar{\alpha}$. Note that we do not assume that α is a vertex of Γ. Assume that α and $\bar{\alpha}$ are both joined by a branch of Γ to a real non-special critical point c of multiplicity 2. Assume furthermore that both branches do not contain letters p, q, r or non-special critical points (if α is a vertex of Γ, this means that α and c are neighbours), and that c is not a root of $\varphi - 1$. Define e (resp. $\bar{e}$) to be the complex edges joining α (resp. $\bar{\alpha}$) to c. Consider a small neighborhood $\mathcal{U}_c$ of c such that $\mathcal{U}_c$ contains both α and $\bar{\alpha}$. Moreover, assume that $\mathcal{U}_c$ does not contain letters r, special points or critical points different from α, $\bar{\alpha}$ and c. We transform Γ into a graph Γ' as in the Figure 4.4. In $\mathcal{U}_\alpha$, the new graph Γ' contains only one vertex β, which is a letter p (resp. q, r, non-special critical point) if so is α (and no other letters p, q, r or vertices). Moreover, the valency of β is equal to two times that of α.

4.3.1.3 The algorithm

The algorithm goes as follows. We achieve conditions **(i)** and **(ii)** first by making transformations of type a). If there is no $\alpha \in \Gamma \setminus \mathbb{R}P^1$ as in Section 4.3.1.2, then the condition **(iii)** is also satified, and we are done. Otherwise, we perform the transformation of type b), this creates one critical point which violates at least one of conditions **(i)** or **(ii)**. Then, we perform a transformation of type a) around this real critical point. Repeating this process sufficiently many times gives us eventually conditions **(i)**, **(ii)** and **(iii)**.

4.3.2 Analysis of dessins d'enfant

In what follows of this section, we assume that φ satisfies conditions **(i)**, **(ii)** and **(iii)**.

Definitions and Notations 4.8. *Define* $I_0 :=]0,1[$*, and denote by the same letter* p_0 *(resp.* q_0*) any root (resp. pole) of* $\phi_{|I_0}$*. Define* $\flat$ *as the number of connected components of the graph of* $\phi_{|I_0}$*, and* $\flat_+$ *as the number of connected components of the graph of* $\phi_{|I_0}$ *situated above the x-axis.*

Remark 4.9. *Note that the functions ϕ and $\varphi = \phi^m$ have the same $\flat$ but not necessarily same $\flat_+$.*

Let S_0 be the total number of roots and poles of $\phi_{|I_0}$.

Lemma 4.10. *We have* $\lfloor \frac{S_0}{2} \rfloor \leq \flat_+ \leq \lfloor \frac{S_0}{2} \rfloor + 1$.

Proof. The roots and poles in I_0 of ϕ are simple, so the sign of ϕ changes when passing through one of them. □

Remark 4.11. *If S_0 is even and $\flat_+ = \frac{|S_0|}{2} + 1$, then the closest branch to 0 (resp. to 1) of the graph of ϕ in I_0 is above the x-axis.*

Note that

$$\phi'(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}H(x)}{Q^2(x)}$$

where $H(x)$ is

$$\alpha P(x)Q(x) + (P'(x)Q(x) - P(x)Q'(x) - (\alpha+\beta)P(x)Q(x))x + (P(x)Q'(x) - P'(x)Q(x))x^2,$$

and thus $\deg H \leq \deg P + \deg Q + 1$. Therefore, since we assumed that all non-special critical points of ϕ are of multiplicity two, the polynomial H has at most $\deg P + \deg Q + 1$ simple roots. One easily computes that ϕ and $\varphi = \phi^m$ have the same set E of non-special critical points (recall that $|E| \leq \deg P + \deg Q + 1$). Moreover, $\phi^{(k)}(x) = 0 \Leftrightarrow (\phi^m)^{(k)}(x) = 0$. Hence a critical point of ϕ with non-zero critical value is a critical point of φ with also non-zero critical value and same multiplicity, and vice versa. Note that if x is a root (simple by assumption) of P (resp. Q), then x is a special point of ϕ^m of multiplicity m, thus corresponds to a vertex of $\Gamma = (\phi^m)^{-1}(\mathbb{R}P^1)$ of valency $2m$.

Set $B = (\phi^{-1}(0, 1, \infty) \cup \{$ non-special critical points $\}) \cap \mathbb{R}$.

Definition 4.12. *A real non-special critical point n is called **useful** if among the two closest points in B, there is a letter r (See Figure 4.5).*

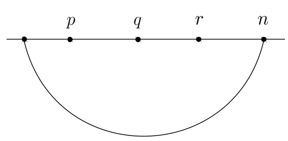

Figure 4.5: The point n is a useful non-special critical point.

Definition 4.13. *Consider two real non-special critical points x_1 and x_2 in I_0 which are neighbours and such that $]x_1, x_2[$ does not contain non-special critical points. Furthermore, consider the disc $\mathcal{D}$ in $\mathbb{C}P^1$ containing $]x_1, x_2[$ with boundary given by the union of the complex arc of Γ joining x_1 to x_2 and its conjugate. Then the **flattening** of Γ with respect to $]x_1, x_2[$ is the dessin d'enfant obtained by collapsing the complex conjugate branches joining x_1 and x_2 to $]x_1, x_2[$ and forgetting all the connected components of Γ contained in $\mathcal{D}$. If there is a letter r in the boundary of $\mathcal{D} \setminus \mathbb{R}P^1$, then this letter and its conjugate are transformed into a single letter $r \in]x_1, x_2[$ (see Figure 4.6).*

Recall that all non-special critical points of ϕ have multiplicity two. In particular, if it is real, such a point has only one neighbor.

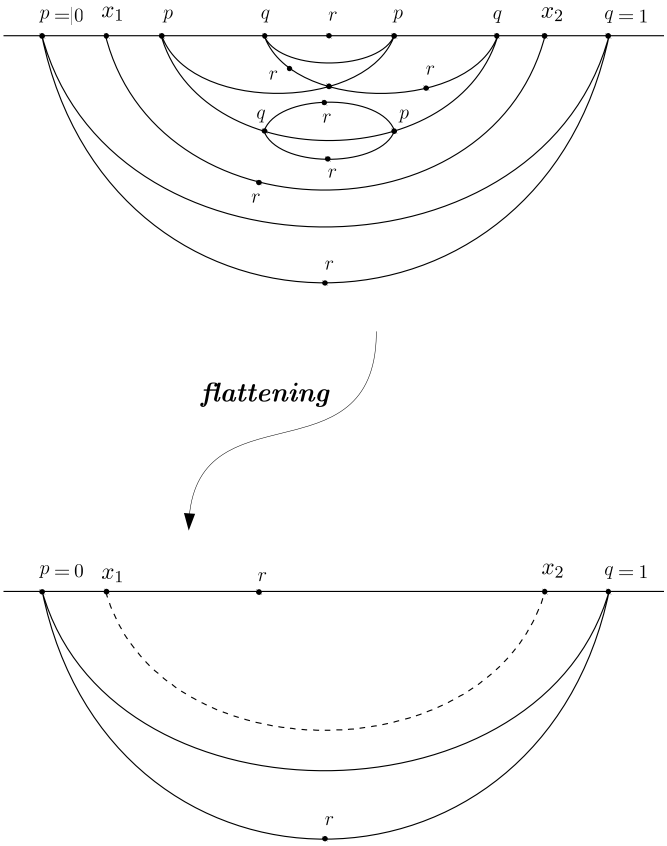

Figure 4.6: Flattening for $m = 2$.

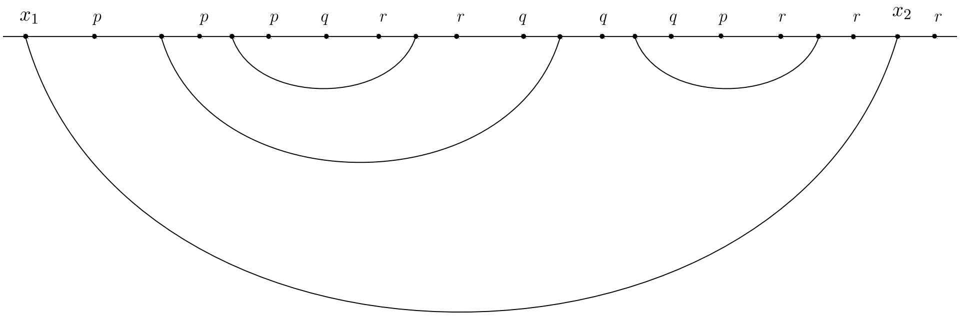

Figure 4.7: In this example, $m = 1$ and $[x_1, x_2]$ contains three useful non-special critical points, four roots and four poles of φ.

Proposition 4.14. *Let x_1, $x_2 \in I_0 =]0,1[$ be two non-special critical points which are neighbours. Assume that all non-special critical points in $]x_1, x_2[$ are neighbours only to each other. Then the number of roots of φ is equal to that of the poles of φ in $]x_1, x_2[$, and this number is bigger than or equal to the number of useful critical points in $[x_1, x_2]$ (See Figure 4.7).*

Proof. Suppose first that $]x_1, x_2[$ does not contain non-special critical points. Then the number of roots (letters p) and poles (letters q) in $]x_1, x_2[$ are equal by the cycle rule (See Figure 4.8).

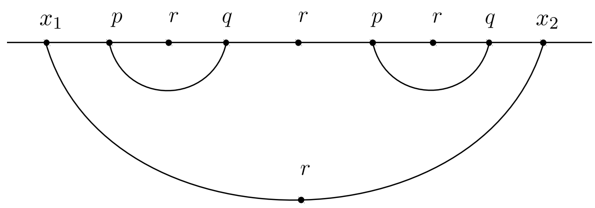

Figure 4.8: The function φ has the same number of roots and poles in $]x_1, x_2[$.

Moreover, x_1 and x_2 cannot both be useful non-special critical points, again since otherwise this contradicts the cycle rule.

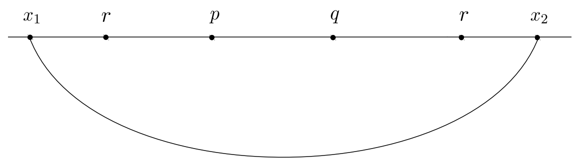

Figure 4.9: Having both non-special critical points useful contradicts the cycle rule.

Assume now that $]x_1, x_2[$ contains non-special critical points. Consider two non-special critical points y_1, $y_2 \in [x_1, x_2]$ which are neighbours and such that $]y_1, y_2[$ does not contain non-special critical points. We have already seen that y_1 and y_2 cannot both be useful, and that $]y_1, y_2[$ contains the same non-zero number of letters p and q. Thus it suffices to prove the result for the dessin d'enfant obtained by flattening Γ with respect to $]y_1, y_2[$. Note that the number of non-special critical points in $]x_1, x_2[$ strictly decreases after such flattening. Therefore, we are reduced to the case where $]x_1, x_2[$ does not contain non-special critical points.

□

Recall that all letters p and q, which are different from 0, 1 or ∞, have the same valency $2m$.

Lemma 4.15. *Let x_1, $x_2 \in I_0$ be critical points which are neighbours and such that $]x_1, x_2[$ does not contain non-special critical points. If one endpoint of $[x_1, x_2]$ is a non-special critical point, then both x_1, x_2 are non-special critical points.*

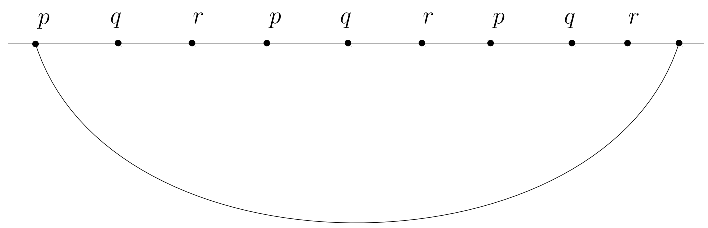

Figure 4.10: No part of Γ can have this configuration given both critical points are in I_0.

Proof. We argue by contradiction. Assume that $]x_1, x_2[=]\tilde{p}, c[$ where c is a non-special critical point and $\tilde{p}$ is a root of P (the case where instead of $\tilde{p}$ we have a root of Q is symmetric). Consider the open disk $\mathcal{D}$ which contains $]\tilde{p}, c[$ and which is bounded by the complex branch of Γ joining $\tilde{p}$ to c together with the conjugate branch. Consider the set of special points in $\mathcal{D} \cup \{\tilde{p}\}$ together with the branches of $\Gamma \cap (\mathcal{D} \cup \tilde{p})$ joining letters p to letters q and not containing any other special points (a branch of Γ is a subset homeomorphic to an interval). This gives a bipartite graph $\mathfrak{G}$. Therefore, the total degree of letters p and the total degree of letters q in $\mathfrak{G}$ are equal. Denote by N_p (resp. N_q) the number of letters p (resp. letters q) contained in $\mathcal{D} \cup \{\tilde{p}\}$. Since $\mathfrak{G}$ is a bipartite graph, we have

$$2mN_q = 2m(N_p - 1) + \deg \tilde{p},$$

where $\deg \tilde{p}$ is the number of branches of $\mathfrak{G}$ adjacent to $\tilde{p}$, and thus we have $1 \leq \deg \tilde{p} \leq 2m - 3$. Therefore $2m(N_p - N_q) = 2m - \deg \tilde{p}$, which is impossible. Indeed, $|2m(N_p - N_q)|$ is either zero or greater than or equal to $2m$, which is not the case for $|2m - \deg \tilde{p}|$. □

Lemma 4.16. *Let α be a non-special critical point in I_0, and $\beta \in \mathbb{R}$ be its neighbor. If β is a root (letter p) or a pole (letter q) of φ, then $\beta \notin I_0$.*

Proof. Assume that $\beta \in \mathbb{R}$ is a root of φ (a letter p), and let us prove that $\beta \notin I_0$ (the case where β is a pole of φ is symmetric). Performing flattening if necessary, we may suppose that the remaining non-special critical points in $[\alpha, \beta]$ are neighbours to special critical points in $[\alpha, \beta]$. Indeed, since non-special critical points cannot be neighbours to complex special critical points. Consider an open interval $J \subset [\alpha, \beta]$ with endpoints a non-special critical point and a special critical point which are neighbours, and such that J does not contain non-special critical points. Note that if $]\alpha, \beta[$ does not contain non-special critical points, then it suffices to consider $J =]\alpha, \beta[$. If $\beta \in I_0$, then the existence of J contradicts Lemma 4.15.

□

By definition, useful critical points of φ have positive critical value. However, when m is even, some of the non-special useful critical points of $\varphi = \phi^m$ may correspond to non-special critical points of ϕ with negative critical value.

Definition 4.17. *A useful critical point x of $\varphi = \phi^m$ is called **positive** if $\phi(x) > 0$.*

These useful positive critical points of ϕ^m will later play a key role via the following Lemma.

Lemma 4.18. *Let U be the set of useful positive non-special critical points in I_0 and let N be the number of solutions of $\phi(x) = 1$ in I_0. Then $N \leq \flat_+ + |U|$.*

Proof. Let C be a connected component of the graph of $\phi_{|I_0}$ situated above the x-axis. Let $I \subset I_0$ be the image of C under the vertical projection. It suffices to prove that in I, the number of solutions of $\phi(x) = 1$ is bounded above by one plus the number of useful positive critical points.

If this number of solutions is zero or one, the bound is trivial. Otherwise, between two consecutive solutions of $\phi(x) = 1$ in I, there is at least one useful positive critical point by Rolle's Theorem.

□

In what follows, by p_1 (resp. q_1) we mean any real root (resp. pole) of φ outside $]0, 1[$.

Lemma 4.19. *Let u_0 and v_0 be two non-special critical points in I_0 which are neighbours to the same point p_1 (resp. q_1). Then the number of useful positive critical points of φ, contained in $[u_0, v_0]$, is less than or equal to one plus half of the total number of roots (letters p) and poles (letters q) of φ in $]u_0, v_0[$.*

Proof. We only prove the result for the point p_1 (the case for q_1 is symmetric). If there are no non-special critical points inside $]u_0, v_0[$, then the result is clear by the cycle rule (see Figure 4.11).

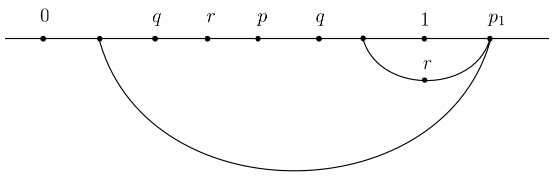

Figure 4.11: An example of a special point outside I_0 that is a neighbor to two non-special critical points in I_0.

Using Proposition 4.14 and flattenings of Γ if necessary, we may assume that $[u_0, v_0]$ does not contain non-special critical points that are neighbours. Then, by Lemma 4.16, the remaining non-special critical points in $[u_0, v_0]$ are neighbours to p_1. Indeed, by condition **(iii)** of (4.3.1), real non-special critical points cannot be neighbours to complex special points. The cycle rule implies that between two consecutive non-special critical points in $[u_0, v_0]$, the total number of special points (letters p, q) is odd. It follows that ϕ takes values of opposite signs at two consecutive non-special critical points in $[u_0, v_0]$. The result follows then as any interval with endpoints two consecutive non-special critical points contains at least one special point. □

Lemma 4.20. *Assume that p_1 (resp. q_1) $\in \{0, 1\}$, and let c be the nearest non-special critical point in I_0 to p_1 (resp. q_1) such that c and p_1 (resp. c and q_1) are neighbours. Then in the open interval I with endpoints c and p_1 (resp. c and q_1), the number of poles (resp. roots) is equal to the number of roots (resp. poles) plus one.*

Proof. We only prove the case for p_1 since the case for q_1 is symmetric. By Proposition 4.14, we only count the remaining special points in I after flattenning Γ with respect to all non-special critical points in I which are neighbours. Note that by Lemma 4.16 and condition **(iii)** of (4.3.1), there do not exist non-special critical points in I after this flattening. Therefore there should be one root between two consecutive poles of ϕ and vice-versa in I. Finally, by the cycle rule, the nearest special points to c and to p_1 in I should both be letters q. □

We now categorize the non-special critical points in I_0 and the special critical points in $\mathbb{R}$.

Definition 4.21. *We first divide the set S_1 of special points outside I_0 in three disjoint subsets:*

• *$S_{1,0}$ (resp. $S_{1,1}$, $S_{1,2}$) is the set of special points in $\mathbb{R}\backslash I_0$ which have no (resp. exactly one, at least two) non-special critical points in I_0 as neighbours.*

Similarly, we divide the set S_0 of special points in I_0 into three disjoint subsets:

• *$S_{0,0}$ is the set of special points in I_0 which are situated between two non-special critical points in I_0 that are neighbours. Note that the points of $S_{0,0}$ are those of S_0 which disappear after flattenings.*
• *$S_{0,2}$ is the set of special points in I_0 which are not in $S_{0,0}$ and which are contained in an interval with two non-special useful critical points that are neighbours of a same point in $S_{1,2}$ (see Figure 4.12).*

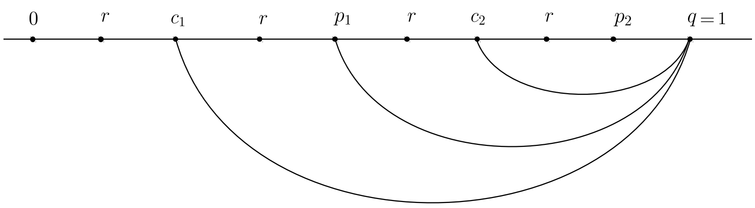

Figure 4.12: A point $q \in S_{1,2}$ and its neighbours: $p_1 \in S_{0,2}$ and two useful critical points c_1 and c_2.

• $S_{0,1} := S_0\backslash(S_{0,0} \cup S_{0,2})$.

Finally, the set U of useful positive critical points in I_0, is divided as follows:

• *$US_{1,1}$ (resp. $US_{1,2}$) is the set of useful positive critical points in I_0 that are neighbours to points of $S_{1,1}$ (resp. $S_{1,2}$).*
• *UN_0 (resp. UN_1) is the set of useful positive critical points in I_0 that are neighbours to non-special critical points in I_0 (resp. outside I_0).*

Remark 4.22. *Note that by definition, we have $|US_{1,1}| \leq |S_{1,1}|$.*

Proposition 4.23. *We have $|US_{1,2}| \leq \frac{|S_{0,2}|}{2} + |S_{1,2}|$ and $|UN_0| \leq \frac{|S_{0,0}|}{2}$.*

Proof. Let us prove the first inequality. Doing flattenings if necessary we may assume that $S_{0,0} = 0$. Then $|US_{1,2}| \leq \frac{|S_{0,2}|}{2} + |S_{1,2}|$ follows directly from Lemma 4.19 applied to each point of $S_{1,2}$ together with the biggest interval $[u_0, v_0]$ such that u_0 and v_0 are non-special critical points which are neighbours to this point in $S_{1,2}$ (see Figure 4.13).

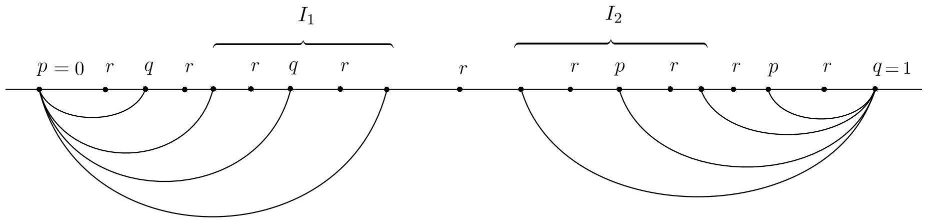

Figure 4.13: There exist elements of $S_{0,2}$ contained in each of I_1 and I_2.

Let us now prove that $|UN_0| \leq \frac{|S_{0,0}|}{2}$. For each point $c \in UN_0$, consider its neighbor $\tilde{c}$ in I_0 ($\tilde{c}$ is a non-special critical point). By Lemma 4.16 and condition **(iii)** of (4.3.1), the non-special critical points of φ between c and $\tilde{c}$ are only neighbours to each other. Applying Proposition 4.14 to each such interval $[\tilde{c}, c]$ (or $[c, \tilde{c}]$) which is maximal in the sense that it is not contained in another interval of the same type (with endpoints a useful positive critical point and a non-special critical point in I_0 which are neighbours), we get $|UN_0| \leq \frac{|S_{0,0}|}{2}$. □

Definition 4.24. *Let Γ be a dessin d'enfant and $x \in \Gamma \cap \mathbb{R}P^1$. A **blowing up** of Γ at x is the new real dessin d'enfant obtained by adding a small circle C in $\mathbb{C}P^1 \setminus \Gamma$ (together with its conjugate $\overline{C}$) which contains x, does not intersect $\Gamma \setminus \{x\}$, and contains letters p, q, r on $C \setminus \{x\}$ such that the cycle rule holds for C and its conjugate (see Figure 4.14). A **blowing down** of a dessin d'enfant is the inverse operation.*

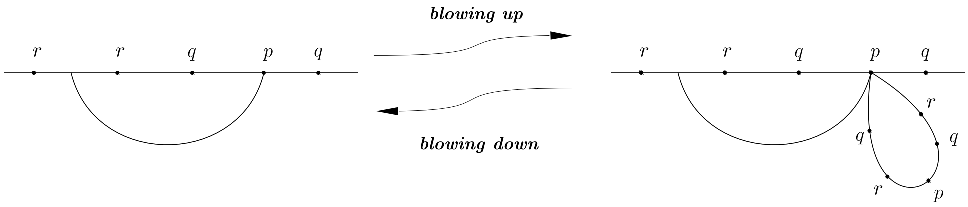

Figure 4.14: The two blowing operations used on a dessin d'enfant.

Lemma 4.25. *Let D be a connected component of $\mathbb{C}P^1 \setminus \Gamma$ such that its boundary ∂D contains at least one real non-special critical point. Then ∂D contains at least two real special points.*

Proof. Consider a connected component of $\partial D \setminus \mathbb{R}P^1$ as in the statement, doing as many blowing-downs as necessary, we may assume that for each connected component C of $\partial D \setminus \mathbb{R}P^1$, we have that $|\partial C| = 2$. Note that $\partial C \subset \mathbb{R}P^1$. Now, by the cycle rule, ∂D contains at least two special points. If two such special points are real, then we are done. Otherwise, there exists a connected component C of $\partial D \setminus \mathbb{R}P^1$ containing a special point of φ. Now from condition **(iii)** of (4.3.1), we get that both points of ∂C are special.

□

Recall that we denote by $H\Gamma$ the union of $\mathbb{R}P^1$ and the intersection of Γ with one component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$.

Definition 4.26. *For any $c \in UN_1$ denote by $\tilde{c}$ its neighbour (a non-special critical point outside I_0) and consider the two connected components of $\mathbb{C}P^1 \setminus H\Gamma$ having the complex arc of $H\Gamma$ joining c to $\tilde{c}$ contained in their boundaries. We will call both boundaries* ***associated cycles*** *to c.*

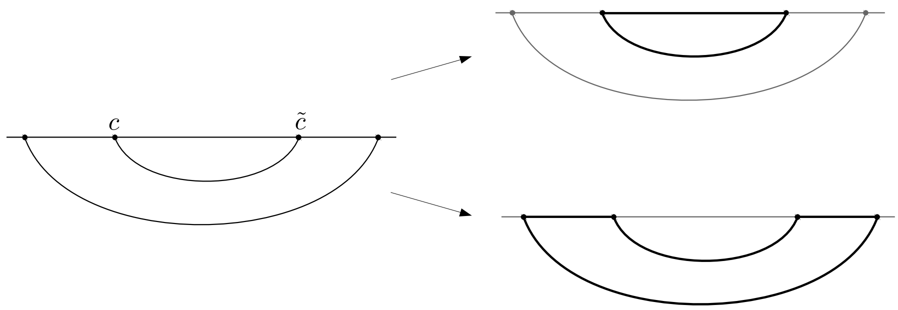

Figure 4.15: The associated cycles to c.

Lemma 4.27. *We have $2|UN_1| \leq |S_{0,1}| + |S_{1,0}|$. Moreover, denoting by k the number of elements of $S_{0,1} \cup S_{1,0}$ which are not contained in cycles associated to some points of UN_1, we have $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - k$. Finally, $2|UN_1| = |S_{0,1}| + |S_{1,0}| - k$ only if any such cycle contains at most two elements of $S_{0,1} \cup S_{1,0}$.*

Proof. Performing flattening if necessary, we may assume without loss of generality that $|S_{0,0}| = 0$. We now show that each cycle ∂D associated to some $c \in UN_1$ contains at least one element of $S_{0,1} \cup S_{1,0}$. Recall that by Lemma 4.25, ∂D contains at least two real special points. We distinguish two cases.

• Assume that $\partial D \cap S_{1,1} \neq \emptyset$. Then by the cycle rule, we get that ∂D also contains at least one letter r (which can be complex) and additional real special points. It is easy to see that none of these additional points belongs to $S_{1,1} \cup S_{1,2}$ (see Figure 4.16). Therefore, ∂D contains at least one element of $S_{0,1} \cup S_{1,0}$.

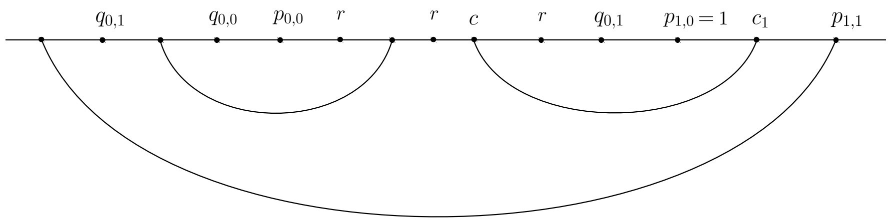

Figure 4.16: The indexes of the letters correspond to those of the sets that contain them. The letter $q_{0,1}$, which is on the left, belongs to one of the associated cycles.

• Assume now that ∂D contains an element of $S_{1,2}$. Then one of the neighbours of this element, which belongs to $\partial D \cap I_0$, is either an element of $S_{0,1}$ or a non-special critical point in I_0. In both cases, reasoning as before, we still obtain that ∂D contains at least one element of $S_{0,1} \cup S_{1,0}$.

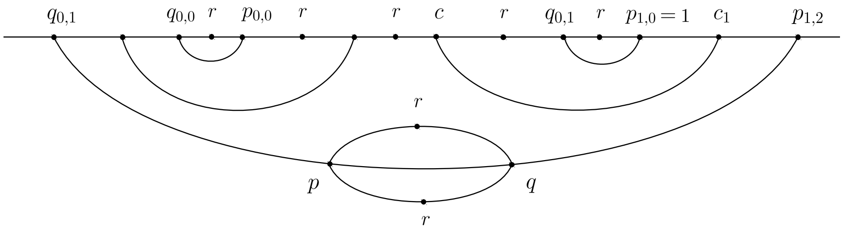

Figure 4.17: The left $q_{0,1}$ is a critical point in the cycle.

We now divide I_0 with respect to the non-special critical points of φ. Let

$$c_1 < c_2 < \cdots < c_N$$

be the non-special critical points of φ in I_0. Consider two consecutive non-special critical points c_i and c_{i+1}.

Assume first that c_i and c_{i+1} belong to UN_1. We show that $]c_i, c_{i+1}[\cup]\tilde{c}_{i+1}, \tilde{c}_i[$, where $\tilde{c}_i$ (resp. $\tilde{c}_{i+1}$) is the neighbor of c_i (resp. c_{i+1}), contains at least two elements of $S_{0,1} \cup S_{1,0}$. Note that $\tilde{c}_i$ and $\tilde{c}_{i+1}$ are non-special critical points outside I_0.

It is easy to see that $]\tilde{c}_{i+1}, \tilde{c}_i[\cap(S_{1,1} \cup S_{1,2}) = \emptyset$. Indeed,$]c_i, c_{i+1}[$ does not contain non-special critical points. Therefore the only special points that can be contained in $]c_i, c_{i+1}[\cup]\tilde{c}_{i+1}, \tilde{c}_i[$ are elements of $S_{0,1} \cup S_{1,0}$, where by Lemma 4.25, there are at least two of them.

Assume now that only one point, say c_i, among c_i and c_{i+1} belongs to UN_1. Then the beginning of the proof shows that the cycle associated to c_i which intersects $[c_i, c_{i+1}]$ contains at least one element of $S_{0,1} \cup S_{1,0}$.

Using again the begining of the proof, we get that the cycle associated to c_1 (resp. c_N) intersecting $[0, c_1]$ (resp. $[c_N, 1]$), contains at least one element of $S_{0,1} \cup S_{1,0}$.

Summing all these inequalities (there is no over-counting), we get $2|UN_1| \leq |S_{0,1}| + |S_{1,0}|$. Furthermore, note that while making this sum, we only consider the points in $S_{0,1} \cup S_{1,0}$ that are contained in the cycles associated to points $c \in UN_1$. Therefore, other points in $S_{0,1} \cup S_{1,0}$ do not contribute to the sum. Denoting their number by k, we get $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - k$. Finally, it is clear from the proof that if $2|UN_1| = |S_{0,1}| + |S_{1,0}| - k$, then any such cycle contains at most two elements of $S_{0,1} \cup S_{1,0}$. □

4.3.3 End of the proof of Theorem 4.2

By Lemma 4.10, Remark 4.22, Proposition 4.23 and Lemma 4.27, we have respectively

$$\flat_+ \leq \frac{|S_0|}{2} + 1, \ |US_{1,1}| \leq |S_{1,1}|, \ |US_{1,2}| \leq \frac{|S_{0,2}|}{2} + |S_{1,2}|, \ |UN_0| \leq \frac{|S_{0,0}|}{2} \tag{4.3.2}$$

$$\text{and} \quad |UN_1| \leq \frac{|S_{0,1}| + |S_{1,0}|}{2}.$$

Moreover, we have $N \leq \flat_+ + |U|$ by Lemma 4.18. Denote by S_c the set of all complex special points of φ.

Note that a root (letter p) or a pole (letter q) of φ can have the value at ∞. Therefore, $|S_0| + |S_1| \leq \deg P + \deg Q + 3 - |S_c|$.

Thus, since $|U| = |UN_0| + |UN_1| + |US_{1,1}| + |US_{1,2}|$, $|S_0| = |S_{0,0}| + |S_{0,1}| + |S_{0,2}|$ and $|S_1| = |S_{1,0}| + |S_{1,1}| + |S_{1,2}|$, we get

$$N \leq |S_0| + |S_1| + 1 - \frac{|S_{1,0}|}{2} - |S_c| \leq \deg P + \deg Q + 4 - \frac{|S_{1,0}|}{2} - |S_c|. \tag{4.3.3}$$

If $|S_{1,0}| > 2$ or $|S_c| > 1$, then by (4.3.3) we have $N \leq \deg P + \deg Q + 2$ and we are done. Note that $|S_c|$ is even since S_c is the set of complex points together with their conjugates. Therefore, let us assume that $|S_{1,0}| \leq 2$ and $|S_c| = 0$. The last equality means that all special points are real and simple.

• Assume that $|S_{1,0}| = 0$. This means that all special points outside I_0 (including 0 and 1) are critical and are neighbours to non-special critical points in I_0. Consider the open interval J_0 (resp. J_1) with endpoints the special point 0 (resp. 1) and a neighbor c_0 (resp. c_1) in I_0 (see Figure 4.18). As a consequence of Lemma 4.20, there exists an odd number of special points in J_0 (resp. J_1). Note that these special points are elements of $S_{0,1}$, and they cannot be contained in any cycle associated to some $c \in UN_1$. Thus, by Lemma 4.27, we have $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - 2$, and therefore we get $N \leq \deg P + \deg Q + 3$.

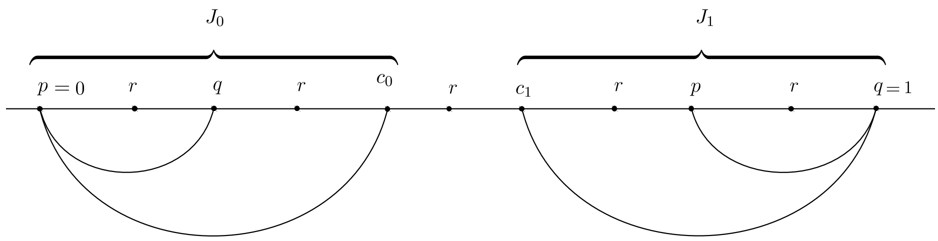

Figure 4.18: Each interval J_0 and J_1 contains an odd number of special points.

We now assume that $N = \deg P + \deg Q + 3$ and prove that this gives a contradiction. Then $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - 2$ and all inequalities in (4.3.2) and (4.3.3) are equalities. In particular, $|S_0|$ is an even number. Then by Remark 4.11 and the fact that there is an odd number of special points in J_0 (resp. J_1), we get that c_0 (resp. c_1) is not a positive useful critical point.

This implies that 0 and 1 do not belong to $S_{1,1}$ (and thus belong to $S_{1,2}$). Indeed, suppose on the contrary that one of 0 or 1, say 0, belongs to $S_{1,1}$. Since c_0 does not belong to $US_{1,1}$, this implies that $|US_{1,1}| < |S_{1,1}|$, a contradiction.

Now, from $0,\ 1 \in S_{1,2}$ it follows that $c_0,\ c_1 \in US_{1,2}$. Denote by $\tilde{c}_0 \in I_0$ the closest non-special critical point to 1 such that $\tilde{c}_0$ is a neighbor to 0, and by K the closed interval with endpoints c_0 and $\tilde{c_0}$. Recall that

$$|US_{1,2}| = \frac{|S_{0,2}|}{2} + |S_{1,2}|, \tag{4.3.4}$$

thus by Lemma 4.19, the number of elements in $K \cap US_{1,2}$ is equal to one plus half the number of elements in $K \cap S_{0,2}$. As c_0 is not a positive useful non-special critical point, if $\tilde{c}_0$ is positive (resp. negative), then $|K \cap S_{0,2}|$ is an odd (resp. even) number, and in both cases we get $|K \cap US_{1,2}|$ is less than one plus half the number of elements in $K \cap S_{0,2}$. This contradicts (4.3.4).

• Assume that $|S_{1,0}| = 1$. This means that there exists only one special point outside I_0 that is not a neighbor to a non-special critical point in I_0. We argue now as in the case $|S_{1,0}| = 0$. We have that at least one special point in $\{0, 1\}$, say 0, is a neighbor to a non-special critical point c_0 in I_0. Then, the interval $J_0 =]0, c_0[$ contains at least one element of $S_{0,1}$ that is not contained in a cycle associated to some point $c \in UN_1$. Thus by Lemma 4.27, we get $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - 1$, and therefore $N \leq \deg P + \deg Q + 3$.

Assume that neither 0 nor 1 belongs to $S_{1,0}$. Then, as discussed in the previous case, since the points 0 and 1 are neighbours to non-special critical points in I_0, we get that at least two elements of $S_{0,1}$ (one in J_0, another one in J_1, see Fig. 4.18) are not contained in a cycle associated to some $c \in UN_1$. Therefore by Lemma 4.27, we get $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - 2$, which yields $N < \deg P + \deg Q + 3$ and we are done.

Assume now that either 0 or 1 belongs to $S_{1,0}$. We assume furthermore that $N = \deg P + \deg Q + 3$ and prove that this gives a contradiction. Using $|S_{1,0}| = 1$, $2|UN_1| \leq |S_{0,1}| + |S_{1,0}| - 1$, $N = \deg P + \deg Q + 3$ and (4.3.3), we get $2|UN_1| = |S_{0,1}| + |S_{1,0}| - 1$ and $|S_0|$ is even. Consider without loss of generality that $0 \in S_{1,0}$. We have $0 \in S_{1,0} \cap \partial D_0$, where ∂D_0 is a cycle associated to some $c_0 \in UN_1$. Indeed, suppose on the contrary that 0 is not contained in a cycle associated to some point $c \in UN_1$. We already saw that there exists an element of $S_{0,1}$ which is not contained in a cycle associated to some $c \in UN_1$. Together with 0 this would give at least two elements of $S_{0,1} \cup S_{1,0}$ that are not contained in such a cycle, and thus $2|UN_1| = |S_{0,1}| + |S_{1,0}| - 2$ by Lemma 4.27. This contradicts $2|UN_1| = |S_{0,1}| + |S_{1,0}| - 1$. Therefore $0 \in S_{1,0} \cap \partial D_0$ where ∂D_0 is a cycle associated to some $c_0 \in UN_1$. By the cycle rule and Lemma 4.25, ∂D_0 contains at least one real special point other than 0. As $|S_{1,0}| = 1$ (and $0 \in S_{1,0}$) these special points can only be elements of $S_{0,1}$. There exists only one special point other than 0 in the interval $]0, c_0[$. Indeed, otherwise ∂D_0 would contain 3 elements of $S_{0,1} \cup S_{1,0}$ which implies $2|UN_1| < |S_{0,1}| + |S_{1,0}| - 1$ (by Lemma 4.27), and thus $N < \deg P + \deg Q + 3$. Now using Remark 4.11, we get that c_0 is not a positive useful critical point, but this contradicts the fact that $c_0 \in UN_1$.

• Assume that $|S_{1,0}| = 2$, then we have $N \leq \deg P + \deg Q + 3$. We assume that $N = \deg P + \deg Q + 3$ and prove that this gives a contradiction. The latter assumption (as discussed in the case $|S_{1,0}| = 0$) means that $|S_0|$ is even and $2|UN_1| = |S_{0,1}| + |S_{1,0}|$ since the inequality in (4.3.3) becomes an equality.

We now show that 0 and 1 are elements of $S_{1,0}$. Assume the contrary, say $0 \notin S_{1,0}$. Then as discussed before (case $|S_{1,0}| = 0$), Lemma 4.20 implies that there exists at least one element of $S_{0,1}$ that is not contained in a cycle associated to some $c \in UN_1$. Therefore by Lemma 4.27, we get $2|UN_1| = |S_{0,1}| + |S_{1,0}| - 1$, a contradiction.

Therefore, the point 0 (resp. 1) belongs to a cycle associated to an element c_0 (resp. c_1) in UN_1. Lemma 4.27 shows that both cycles contain at most one element of $S_{0,1}$ each, since otherwise $2|UN_1| < |S_{0,1}| + |S_{1,0}|$. However, as discussed before (using Remark 4.11), this implies that c_0 and c_1 are not positive useful critical points, a contradiction.

4.4 The case of two trinomials: proof of Theorem 4.3

It is shown in [LRW03] that the maximal number of positive solutions of a system of two trinomial equations in two variables is five. In this section, we prove Theorem 4.3. We recall the proof of Theorem 4.1 in this special case in order to describe what happens in terms of the dessin d'enfant Γ when the maximal number five of positive solutions is reached. Consider a system

$$c_0 \cdot u^{w_0} + c_1 \cdot u^{w_1} + c_2 \cdot u^{w_2} = c_3 \cdot u^{w_3} + c_4 \cdot u^{w_4} + c_5 \cdot u^{w_5} = 0 \qquad (4.4.1)$$

where all $c_i \in \mathbb{R}^*$, $u = (u_1, u_2) \in \mathbb{R}^2$ and all $w_i \in \mathbb{Z}^2$.

Lemma 4.28. *If a facet e_1 of the Newton triangle of the first equation and a facet e_2 of the Newton triangle of the second equation are parallel, then* (4.4.1) *has strictly less than five positive solutions.*

Proof. Assume that the Newton triangles of (4.4.1) satisfies the conditions of the lemma. Suppose without loss of generality that the parallel facets e_1 and e_2 are the convex hulls of the supports of the truncated binomials $c_0u^{w_0} + c_1u^{w_1}$ and $c_3u^{w_5} + c_4u^{w_4}$. We may assume without loss of generality that $w_0 = w_5 = 0$ and $c_0 = c_5 = 1$. Performing a monomial change of coordinates as in the beginning of Section 4.2 if necessary, we may also assume that $|c_1| = |c_2| = 1$. The system

$$\begin{aligned} y &= \epsilon_0 + \epsilon_1 x, \\ 1 &+ c_3x^{m_3} + c_4x^{m_4}y^{n_4} = 0, \end{aligned} \qquad (4.4.2)$$

with $\epsilon_0, \epsilon_1 \in \{-1, +1\}$ and all $m_3, m_4, n_4 \in \mathbb{Q}$, has the same number of non-degenerate positive solutions as (4.4.1). Indeed, the system (6.1.1) is obtained from (4.4.1) by making the monomial change of coordinates $(u_1, u_2) \mapsto (x, y)$ defined by $x = u^{w_1}$ and $y = u^{w_2}$ which preserves the number of positive solutions.

Therefore, the number of positive solutions of (6.1.1) is equal to the number of positive solutions in $I_{\epsilon_0,\epsilon_1}$ of $f(x) = 0$, where

$$f(x) = 1 + a_3x^{m_3} + a_4x^{m_4}(\epsilon_1 x + \epsilon_0)^{n_4}$$

and $I_{\epsilon_0,\epsilon_1} = \{x \in \mathbb{R}_{>0} \mid \epsilon_0 + \epsilon_1 x > 0\}$. Since f has no poles in $I_{\epsilon_0,\epsilon_1}$, by Rolle's Theorem, if $f(x) = 0$ has five positive solutions in $I_{\epsilon_0,\epsilon_1}$ then $f'(x) = 0$ has four positive solutions in the same interval. We prove Lemma 4.28 by showing that the number of positive roots of f' in $I_{\epsilon_0,\epsilon_1}$ is less or equal to 3. Making similar computations as above (at the beginning of this section), we obtain $f'(x) = 0 \Leftrightarrow \phi(x) = 1$, where

$$\phi(x) = x^{m_4 - m_3}(\epsilon_0 + \epsilon_1 x)^{n_4 - 1}\rho(x)$$

and $\deg \rho = 1$.

Note that the result becomes trivial if $\epsilon_0 = \epsilon_2 = -1$ since the first equation of (6.1.1) has no positive solutions. Therefore, we consider three cases.

- First case: $\epsilon_0 = 1$ and $\epsilon_1 = -1$. Then $I_{\epsilon_0,\epsilon_1} =]0, 1[$, and the result comes directly from Theorem (4.2) applied to ϕ.

- Second case: $\epsilon_0 = -1$ and $\epsilon_1 = 1$. Then $I_{\epsilon_0,\epsilon_1} =]1, +\infty[$, and we consider the function $\tilde{\phi}(x) = \phi(1/x)$. Then $\sharp\{x \in]1, +\infty[\mid \phi(x) = 1\} = \sharp\{x \in]0, 1[\mid \tilde{\phi}(x) = 1\}$, and the result follows by applying Theorem (4.2) to

  $$\tilde{\phi}(x) = x^{m_3 - m_4 - n_4}(1 - x)^{n_4 - 1}\tilde{\rho}(x),$$

  with $\deg \tilde{\rho} = 1$.

- Third case: $\epsilon_0 = 1$ and $\epsilon_1 = 1$. Then, ϕ has at most one pole in $\mathbb{R}_{>0}$ (a root of ρ). Similarly to the proof of Lemma 4.4, we have

$$\phi'(x) = x^{m_4 - m_3 - 1}(1+x)^{n_4 - 2}h_2(x),$$

where h_2 is a polynomial of degree at most 2, thus ϕ' has at most two roots. Therefore, the result comes as a consequence of Rolle's Theorem and by noting that the changes of sign (if they exist) of ϕ in $\mathbb{R}_{>0}$ occur only at a root of ρ.

□

Remark 4.29. *Note that as a consequence of Lemma 4.28, we retrieve the fact that if a system* (4.4.1) *has five positive solutions, then the Minkowski sum of the Newton triangles associated to each equation of* (4.4.1) *is an hexagon [LRW03].*

In what follows, we assume that the support of each equation of (4.4.1) is non-degenerate i.e. it is not contained in a line. Furthermore, we suppose that the system has positive solutions, thus the coefficients of each equation of (4.4.1) have different signs. Therefore without loss of generality, let $c_0 = -1$, $c_1 \cdot c_2 < 0$, $c_5 = -1$, $c_3 > 0$ and $c_4 > 0$.

Since we are looking for solutions of (4.4.1) with non-zero coordinates, one can assume that $w_0 = w_5 = (0,0)$. Let k_3 be the greatest common divisor of the coordinates of w_3. Setting $z = c_3 \cdot u^{\frac{w_3}{k_3}}$ and choosing any basis of $\mathbb{Z}^2$ with first vector $\frac{1}{k_3} \cdot w_3$, we get a monomial change of coordinates $(u_1, u_2) \mapsto (z, w)$ of $(\mathbb{C}^*)^2$ such that $c_3 \cdot u^{w_3} = z^{k_3}$ and $c_4 \cdot u^{w_4} = z^{k_4} w^{l_4}$. Replacing w by w^{-1} if necessary, we assume that $l_4 > 0$. Indeed, $l_4 \neq 0$, since by assumption, the support of each equation of (4.4.1) is non-degenerate. With respect to these new coordinates, the system (4.4.1) becomes the polynomial system

$$-1 + a_1 \cdot z^{k_1} w^{l_1} + a_2 \cdot z^{k_2} w^{l_2} = -1 + z^{k_3} + z^{k_4} w^{l_4} = 0 \tag{4.4.3}$$

where a_i has the same sign of c_i for $i = 1, 2$. Note that since c_3 and c_4 are positive, (4.4.1) and (4.4.3) have the same number of positive solutions.

We now look for the positive solutions of (4.4.3). The second equation of this system may be written as $w = x^{\alpha}(1-x)^{\beta}$, where $x := z^{k_3}$, $\alpha = -k_4/(k_3 l_4)$ and $\beta = 1/l_4$. It is clear that since $z, w > 0$, we have $x \in I_0 =]0,1[$. Plugging z and w in the first equation of 4.4.3, we get

$$-1 + a_1 \cdot x^{\alpha_1}(1-x)^{\beta_1} + a_2 \cdot x^{\alpha_2}(1-x)^{\beta_2} = 0, \tag{4.4.4}$$

where $\alpha_i := \dfrac{k_i l_4 - k_4 l_i}{k_3 l_4}$ and $\beta_i := \dfrac{l_i}{l_4}$ for $i = 1, 2$. The number of positive solutions of (4.4.1) is equal to the number of solutions of (4.4.4) in I_0. Therefore we want to bound the number of solutions in I_0 of $f(x) = 1$ where

$$f(x) := a_1 \cdot x^{\alpha_1}(1-x)^{\beta_1} + a_2 \cdot x^{\alpha_2}(1-x)^{\beta_2}. \tag{4.4.5}$$

Note that the function f has no poles in I_0, thus by Rolle's theorem we have $\sharp\{x \in I_0 \mid f(x) = 1\} \leq \sharp\{x \in I_0 \mid f'(x) = 0\} + 1$. Since

$$f'(x) = a_1 x^{\alpha_1 - 1}(1-x)^{\beta_1 - 1}\rho_1(x) + a_2 x^{\alpha_2 - 1}(1-x)^{\beta_2 - 1}\rho_2(x),$$

where $\rho_i(x) = \alpha_i - (\alpha_i + \beta_i)x$ for $i = 1, 2$, we get $f'(x) = 0 \Leftrightarrow \phi(x) = 1$, where

$$\phi(x) = -\frac{a_1}{a_2} \cdot \frac{x^{\alpha_1 - \alpha_2}(1-x)^{\beta_1 - \beta_2}\rho_1(x)}{\rho_2(x)}.$$

Thus applying Theorem 4.2 (with $\deg \rho_1 = \deg \rho_2 = 1$) we get $\sharp\{x \in I_0 \mid f'(x) = 0\} \leq 4$, and therefore $\mathcal{S}(3,3) \leq 5$.

We now start the proof of Theorem 4.3. The property that Δ_1 and Δ_2 do not alternate is preserved under monomial change of coordinates. Thus it suffices to prove Theorem 4.3 for the system (4.4.3). As we just saw before, if (4.4.3) has five positive solutions, then $\phi(x) = 1$ has four solutions in I_0. We look for necessary conditions on the dessin d'enfant $\Gamma = (\phi^m)^{-1}(\mathbb{R}P^1)$ (where m is a natural integer such that $\varphi = \phi^m$ is a rational function as in the previous section). More precisely, we want to know the positions of the root $\tilde{p} = \frac{\alpha_1}{\alpha_1+\beta_1}$ and the pole $\tilde{q} = \frac{\alpha_2}{\alpha_2+\beta_2}$ of φ relatively to 0 and 1 in $\mathbb{R}P^1$.

The **normal fan** of a n-dimensional convex polytope in $\mathbb{R}^n$ is the complete fan with one-dimensional cones directed by the outward normal vectors of the $(n-1)$-faces of this polytope. Denote by Δ_1 and Δ_2 the Newton polytopes of the first and the second equation of (4.4.3) respectively.

Definition 4.30. *Let Δ_1 and Δ_2 be two 2-dimensional polygons in $\mathbb{R}^2$ with the same number of edges. In other words, their respective normal fans $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same numbers of 1-cones and 2-cones respectively. We say that Δ_1 and Δ_2* ***alternate*** *if every 2-cone of $\mathcal{F}_2$ contains properly a 1-cone of $\mathcal{F}_1$ (properly means that the origin is the only common face), see Figure 4.19.*

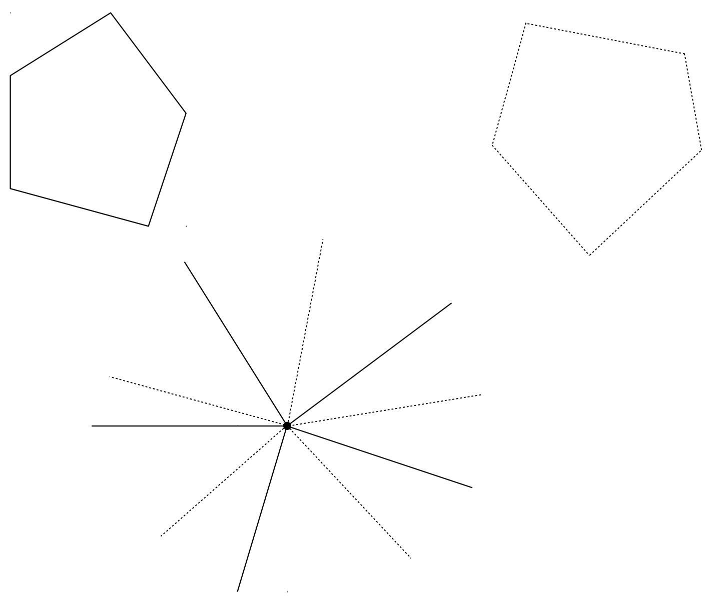

Figure 4.19: Two polytopes that alternate.

Another example that illustrates Theorem 4.3 (where Δ_1 and Δ_2 do not alternate) is the system

$$x^5 - (49/95)x^3y + y^6 = y^5 - (49/95)xy^3 + x^6 = 0, \tag{4.4.6}$$

taken from [Roj] that has five positive solutions.

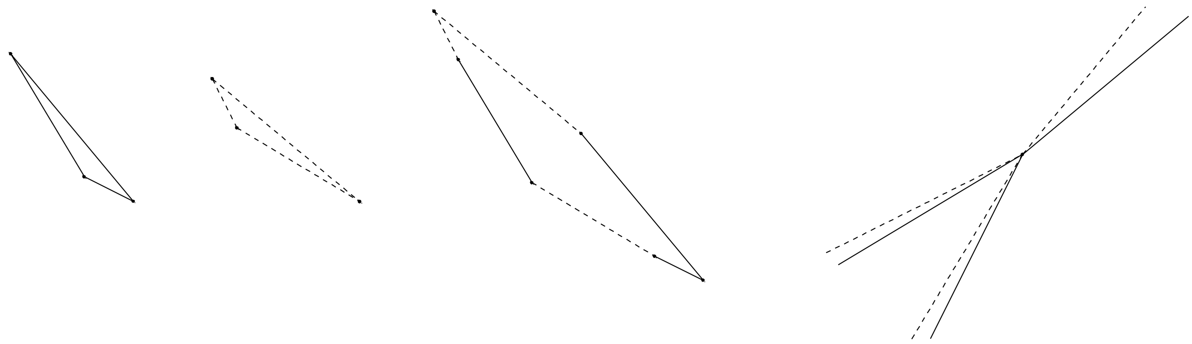

Figure 4.20: The Newton polytopes, their Minkowski sum and the associated normal fans of (4.4.6).

Recall that $k_3 > 0$ and $l_4 > 0$. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) denote the normal fan of Δ_1 (resp. Δ_2). The polygon Δ_2 together with $\mathcal{F}_2$ are represented in Figure 4.21. The outward normal vectors of the three edges of Δ_2 are the vectors $F_{0,3} = (0, -k_3)$, $F_{0,4} = (-l_4, k_4)$ and $F_{3,4} = (l_4, k_3 - k_4)$. The one-dimensional cones of $\mathcal{F}_1$ are generated by vectors $F_{0,1} = \epsilon_{01}(-l_1, k_1)$, $F_{0,2} = \epsilon_{02}(-l_2, k_2)$ and $F_{1,2} = \epsilon_{12}(l_1 - l_2, k_2 - k_1)$, where $\epsilon_{ij} \in \{\pm 1\}$.

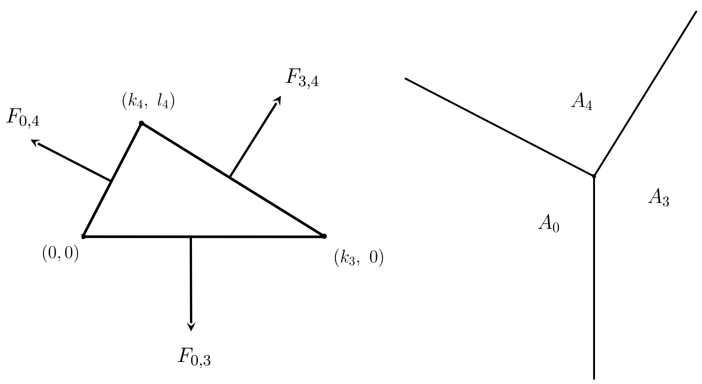

Figure 4.21: The triangleΔ_2 and its normal fan $\mathcal{F}_2$.

Recall that α_1 and α_2 (resp. β_1 and β_2) are the powers of x (resp. $1 - x$) appearing in (4.4.4).

Lemma 4.31. *If* (4.4.3) *has five positive solutions, then we have the following conditions*

$$\alpha_1 - \alpha_2 \neq \beta_2 - \beta_1,\ \alpha_1 \neq \alpha_2,\ \beta_1 \neq \beta_2,\ \alpha_i + \beta_i \neq 0,\ \alpha_i \neq 0 \text{ and } \beta_i \neq 0 \qquad \text{for} \quad i = 1, 2.$$

Proof. Consider any two normal vectors of Δ_1 and Δ_2 each, if they are collinear, then by Lemma 4.28, the system (4.4.3) has strictly less than five positive solutions. We now proceed by contradiction.

Assume that $\alpha_1 - \alpha_2 = \beta_2 - \beta_1$. Then we have

$$\frac{k_1l_4 - k_4l_1 - k_2l_4 + k_4l_2}{k_3l_4} = \frac{k_3l_2 - k_3l_1}{k_3l_4} \Rightarrow (l_1 - l_2)(k_3 - k_4) + l_4(k_1 - k_2) = 0,$$

thus the wedge product $F_{3,4} \wedge F_{1,2}$ vanishes, a contradiction. Similarly, if $\alpha_1 = \alpha_2$ (resp. $\beta_1 = \beta_2$), then we get

$$\frac{k_1l_4 - k_4l_1}{k_3l_4} = \frac{k_2l_4 - k_4l_2}{k_3l_4} \Rightarrow k_4(l_2 - l_1) - l_4(k_2 - k_1) = 0$$

(resp. $k_3(l_1 - l_2) = 0$) and thus $F_{0,4} \wedge F_{1,2} = 0$ (resp. $F_{0,3} \wedge F_{1,2} = 0$), a contradiction. Let $i \in \{1, 2\}$. Using the same arguments, if $\alpha_i = 0$, $\beta_i = 0$ or $\alpha_i = -\beta_i$, we get

$$k_il_4 - k_4l_i = 0 \Rightarrow F_{0,4} \wedge F_{0,i} = 0,$$

$$l_i = 0 \Rightarrow F_{0,3} \wedge F_{0,i} = 0 \quad \text{or}$$

$$\frac{k_il_4 - k_4l_i}{k_3l_4} = \frac{k_3l_i}{k_3l_4} \Rightarrow k_il_4 - l_i(k_4 - k_3) = 0 \Rightarrow F_{3,4} \wedge F_{0,i} = 0$$

respectively, and in each of these cases this is a contradiction. □

Corollary 4.32. *If* (4.4.3) *has five positive solutions, then 0 (resp. 1, ∞) is a special point of φ and $\tilde{p}$ (resp. $\tilde{q}$) does not belong to $\{0, 1, \infty\}$.*

Without loss of generality, we assume that $\alpha_1 > \alpha_2$ considering φ^{-1} instead of φ if necessary. The following key result will play an important role in relating the arrangment of the special points of φ and the faces of $\Delta_1 + \Delta_2$.

Proposition 4.33. *Assume that $\sharp\{x \in I_0 |\ \phi(x) = 1\} = 4$. If $\beta_1 > \beta_2$, then*

$$\frac{\alpha_1}{\alpha_1 + \beta_1} < \frac{\alpha_2}{\alpha_2 + \beta_2} < 0 \quad \text{or} \quad 1 < \frac{\alpha_2}{\alpha_2 + \beta_2} < \frac{\alpha_1}{\alpha_1 + \beta_1}.$$

And if $\beta_1 < \beta_2$, then

$$0 < \frac{\alpha_2}{\alpha_2 + \beta_2} < 1 < \frac{\alpha_1}{\alpha_1 + \beta_1} \quad \text{or} \quad \frac{\alpha_2}{\alpha_2 + \beta_2} < 0 < \frac{\alpha_1}{\alpha_1 + \beta_1} < 1.$$

Before giving the proof of Proposition 4.33, we need an intermediate result. Assume that $\phi(x) = 1$ has four solutions in I_0 and consider the open interval $\tilde{I}$ with endpoints $\tilde{p}$ and $\tilde{q}$. Recall that we have $a_1 \cdot a_2 < 0$. Therefore the sign of $\phi(x)$ in I_0 is the same as that of

$$\frac{\rho_1(x)}{\rho_2(x)}.$$

Thus the solutions of $\phi(x) = 1$ are either all inside or outside $\tilde{I}$. Indeed, the sign of ϕ changes when passing through $\tilde{p}$ (resp. $\tilde{q}$). Note that $\tilde{p} \neq \tilde{q}$, because otherwise we get $\phi(x) = kx^{\alpha_1 - \alpha_2}(1-x)^{\beta_1 - \beta_2}$ for some $k \in \mathbb{R}$, which would imply that the equation $\varphi = \phi^m(x) = 1$ has at most two solutions in I_0.

Lemma 4.34. *We have $\tilde{I} \not\subseteq I_0$ and $I_0 \not\subseteq \tilde{I}$.*

Proof. We argue by contradiction. First, assume that $\tilde{I} \subset I_0$. Denote by J_0 (resp. J_1) the left (resp. right) connected component of $I_0 \setminus \tilde{I}$. Three cases exist.

1. Assume that all four solutions (letter r) of $\varphi(x) = 1$ are contained in $\tilde{I}$. Then by Rolle's theorem, there exists at least three non-special critical points of φ in $\tilde{I}$. Recall that φ has at most three non-special critical points, this means that all non-special critical points of φ are contained in $\tilde{I}$. Furthermore, we have $\alpha_1 > \alpha_2$, so 0 is a root (letter p) of φ, and thus $\tilde{q} < \tilde{p}$, which implies that 1 is a pole (letter q) of φ. In this case, if ∞ is a root (resp. pole) of φ (recall that by Corollary 4.32, ∞ is either a root or a pole of φ), then there exists a non-special critical point that is smaller than 0 (resp. bigger than 1). This gives a contradiction.

2. Assume that the four solutions of $\varphi(x) = 1$ in I_0 belong to J_0 (the case where the roots are in J_1 is symmetric). Then by Rolle's theorem, all non-special critical points of φ (recall that it has at most three non-special critical points) are contained in J_0. As a consequence of Lemma 4.20, we get that none of these non-special critical points can be neighbours to the special point 0 or 1. Moreover, by Lemma 4.16, these non-special critical points cannot be neighbours to $\tilde{p}$ or $\tilde{q}$. The cycle rule shows that the non-special critical points in J_0 cannot be neighbours to each other. We conclude that the only possible neighbor of each non-special critical point in J_0 is the point ∞. This contradicts the cycle rule.

3. Assume that at least one solution of $\varphi(x) = \phi^m(x) = 1$ is contained in J_0 and at least another one is contained in J_1. Thus, in particular all four solutions of $\phi(x) = 1$ belong to $J_0 \cup J_1$ (since they are all either inside or outside $\tilde{I}$). Then by Rolle's theorem, there exist at least two non-special critical points of φ contained in $J_0 \cup J_1$. Therefore, the interval $\tilde{I}$ does not contain non-special critical points since $\tilde{I}$ can only contain an even number of non-special critical points. As a consequence of Lemma 4.20, these non-special critical points cannot be neighbours to special points 0 or 1, and by Lemma 4.16, they cannot be neighbours to $\tilde{p}$ or $\tilde{q}$.

   We now prove that non-special critical points in $J_0 \cup J_1$ cannot be neighbours. Indeed, assume on the contrary, that there exists a non-special critical point $c \in I_0$ that is a neighbor to a non-special critical point $\tilde{c} \in I_0$. Then both c and $\tilde{c}$ cannot be contained in the same interval J_0 or J_1, otherwise this will contradict the cycle rule. Assume without loss of generality that $c \in J_0$ and $\tilde{c} \in J_1$. Recall that φ has at most three non-special critical points in I_0. By Proposition 4.14, among c and $\tilde{c}$, one of them, say c, is not useful. We show that c is the only non-special critical point of φ contained in J_0. Assume that there exists a non-special critical point in J_0 other than c. Then, as c is not useful, J_0 will contain at most one letter r. Moreover, $\tilde{c}$ is the only non-special critical point in J_1, and thus J_1 contains at most two solutions of $\phi(x) = 1$. Therefore the total number of solutions of $\phi(x) = 1$ in $J_0 \cup J_1$, and thus in I_0, can be at most three, a contradiction. We have proved that c is the only non-special critical point of φ contained in J_0. Note that as J_0 contains only one non-special critical point, which is not useful, we have that J_0 does not contain solutions of $\phi(x) = 1$. Finally, since J_1 has at most two non-special critical points, it has at most three solutions of $\phi(x) = 1$. As before, we get that $\phi(x) = 1$ has at most three solutions in I_0, a contradiction. We have finished to prove that non-special critical points in $J_0 \cup J_1$ cannot be neighbours.

   We now prove that non-special critical points in $J_0 \cup J_1$ cannot be neighbours to non-special critical points outside I_0. Arguing by contradiction, assume that there exists a non-special critical point $c_0 \in J_0 \cup J_1$ that is a neighbor to a non-special critical point $c_1 \notin I_0$. Then, as $\tilde{p}$ and $\tilde{q}$ are inside I_0, the number of special critical points in the open interval K, with endpoints c_0 and c_1, contains an odd number of special points among 0, $\tilde{p}$, $\tilde{q}$ and 1. Note that there do not exist non-special critical points in $K \setminus I_0$. Indeed, otherwise c_0 would be the only non-special critical point of φ in I_0, which would contradict the fact that $\phi(x) = 1$ has four solutions in I_0. Also there is no non-special critical points in $K \cap I_0$. Indeed, otherwise there would be only one such point in $K \cap I_0$, which obviously is not a neighbor of c_0 or c_1. Moreover, this non-special critical point in $K \cap I_0$ is not a neighbor to $\tilde{p}$ or $\tilde{q}$ by Lemma 4.16, and not a neighbor to 0 or 1 by Lemma 4.20. This shows that there cannot be a non-special critical point $K \cap I_0$. The odd number of special points in K cannot be equal to one since this would contradict the cycle rule. Thus this number is equal to three. Consider the closed disc $\mathfrak{D}$ in $\mathbb{C}P^1$ with boundary given by the union of K and a complex arc of Γ joining c_0 to c_1. Note that K contains either two roots and one pole of φ, or two poles and one root of φ. Moreover, K does not contain non-special critical points of φ. It follows that the cycle rule is violated inside $\mathfrak{D}$.

To sum up, there are at least two non-special critical points in $J_0 \cup J_1$. We showed that they are not neighbours to 0, 1, $\tilde{p}$, $\tilde{q}$ or other non-special critical points. Moreover, it is obvious that they cannot be all neighbours to ∞ by the cycle rule, thus we get a contradiction.

We have finished to prove that $\tilde{I} \not\subseteq I_0$, and now we prove that $I_0 \not\subseteq \tilde{I}$. Assume on the contrary that $I_0 \subset \tilde{I}$. We have 4 solutions of $\phi(x) = 1$ in I_0, so by Rolle's theorem, all three non-special critical points of ϕ are in I_0. This implies that $\tilde{q} < 0$ and $\tilde{p} > 1$. Indeed, 0 is a root of ϕ (since $\alpha_1 > \alpha_2$), and there is no non-special critical points in $\tilde{I} \setminus I_0$. Recall that by Corollary 4.32, the value ∞ is either a root or a pole of φ. If ∞ is a root (resp. pole) of φ, then by Rolle's theorem, there should be a non-special critical point between $\tilde{p}$ (or $\tilde{q}$) and ∞, a contradiction. □

4.4.1 Proof of Proposition 4.33

By Lemma 4.34, we either have that $I_0 \cap \tilde{I} = \emptyset$ or that only one endpoint of $\tilde{I}$ is contained in I_0.

Assume first that only one endpoint $\tilde{e}$ of $\tilde{I}$ belongs to I_0. We already saw that the four solutions of $\phi(x) = 1$ in I_0 are all either inside or outside $\tilde{I}$. Therefore these four solutions, and thus all three non-special critical points of φ, are all either bigger or smaller than $\tilde{e}$. Recall that 0 is a root of φ.

- Assume that all four solutions of $\phi(x) = 1$ are bigger than $\tilde{e}$. Then, as shown at the top of Figure 4.22, $\tilde{e}$ is equal to $\tilde{q}$, and thus $\tilde{p}$ belongs to $]1, \infty[$, since otherwise this would give a non-special critical point smaller than $\tilde{e}$. It follows that 1 is a pole of φ, which means that $\beta_1 < \beta_2$. Moreover, we get that $0 < \frac{\alpha_2}{\alpha_2+\beta_2} < 1 < \frac{\alpha_1}{\alpha_1+\beta_1}$.
- Assume now that all four solutions of $\phi(x) = 1$ are smaller than $\tilde{e}$. Then, as shown at the bottom of Figure 4.22, $\tilde{e}$ is equal to $\tilde{p}$, and thus $\tilde{q}$ belongs to $]\infty, 0[$, since otherwise this would give a non-special critical point bigger than $\tilde{e}$. It follows again that 1 is a pole of φ, which means that $\beta_1 < \beta_2$. Moreover, we get that $\frac{\alpha_2}{\alpha_2+\beta_2} < 0 < \frac{\alpha_1}{\alpha_1+\beta_1} < 1$.

Assume now that $\tilde{I} \cap I_0 = \emptyset$. Recall that by Rolle's theorem, all three non-special critical points of φ are contained in I_0.

- Assume that both $\tilde{p}$ and $\tilde{q}$ are negative. Since 0 is a root of φ, we have $\tilde{p} < \tilde{q} < 0$ i.e. $\frac{\alpha_1}{\alpha_1+\beta_1} < \frac{\alpha_2}{\alpha_2+\beta_2} < 0$. Therefore 1 is a root of φ, which means $\beta_1 > \beta_2$ (See top of Figure 4.23).

- Assume that both $\tilde{p}$ and $\tilde{q}$ are bigger than 1. Since 0 is a root of φ, we have that ∞ is a pole, and thus $1 < \tilde{q} < \tilde{p}$, i.e. $1 < \frac{\alpha_2}{\alpha_2+\beta_2} < \frac{\alpha_1}{\alpha_1+\beta_1}$. Therefore 1 is a root of φ, which means that $\beta_1 > \beta_2$ (See bottom of Figure 4.23).

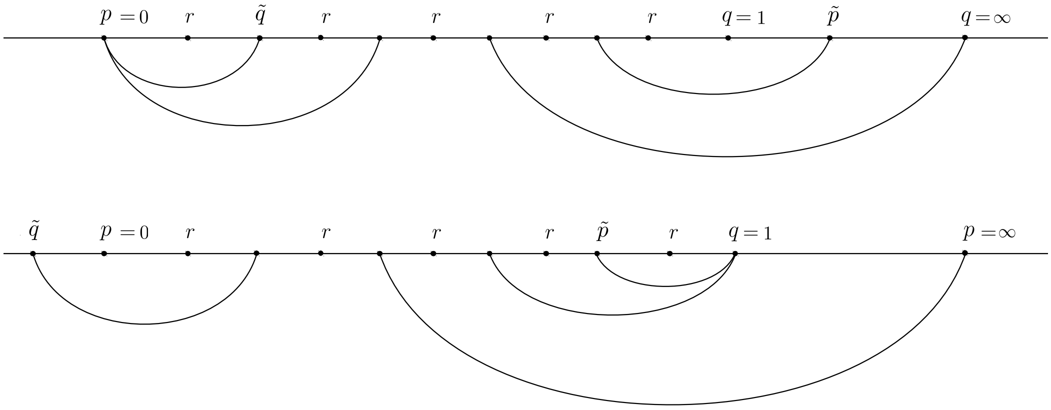

Figure 4.22: At the top: $0 < \tilde{q} < 1 < \tilde{p}$. At the bottom: $\tilde{q} < 0 < \tilde{p} < 1$

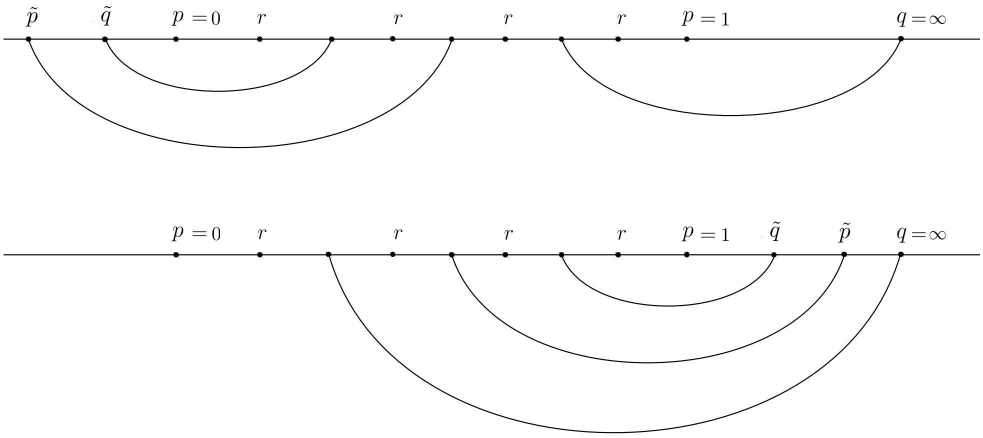

Figure 4.23: At the top: $\tilde{p} < \tilde{q} < 0$. At the bottom: $1 < \tilde{q} < \tilde{p}$

4.4.2 End of proof of Theorem 4.3

Assume that $\phi(x) = 1$ has 4 solutions in I_0. We prove that Δ_1 and Δ_2 do not alternate by looking at each of the four cases of conditions presented in Proposition 4.33. We prove that in each case, there exists a 2-cone A_i of the fan $\mathcal{F}_2$, that does not contain any 1-cone of $\mathcal{F}_1$. In order to do that, we look at the signs of the wedge products of the generators of the 1-cones of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Recall that

$$\tilde{p} = \frac{\alpha_1}{\alpha_1+\beta_1}, \quad \tilde{q} = \frac{\alpha_2}{\alpha_2+\beta_2}, \ \alpha_1 > \alpha_2, \quad \text{and} \quad k_3, l_4 > 0,$$

and for $i = 1, 2$, we have

$$\alpha_i = \frac{k_i l_4 - k_4 l_i}{k_3 l_4} \quad \text{and} \quad \beta_i := \frac{l_i}{l_4}. \tag{4.4.7}$$

• Assume that $\beta_1 < \beta_2$ and $0 < \tilde{q} < 1 < \tilde{p}$. From the proof of Proposition 4.33, we know that the roots of $\phi(x) = 1$ are inside $]\tilde{q}, \tilde{p}[$, thus $(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) < 0$ since $\rho_i(x) = \alpha_i - (\alpha_i + \beta_i)x$ for $i = 1, 2$. The fact that both $\tilde{p}$ and $\tilde{q}$ are positive implies that $\alpha_1(\alpha_1 + \beta_1) > 0$ and $\alpha_2(\alpha_2 + \beta_2) > 0$. Consequently, we have $\alpha_1\alpha_2 < 0$. Furthermore, as $\alpha_1 > \alpha_2$, we have $\alpha_2 < 0 < \alpha_1$. From $\alpha_2 < 0$ and $\frac{\alpha_2}{\alpha_2+\beta_2} > 0$, we get $\alpha_2 + \beta_2 < 0$ and thus $\alpha_2 + \beta_2 < 0 < \alpha_1 + \beta_1$. Furthermore, as $\alpha_2 + \beta_2 < 0$ (resp. $\alpha_1 + \beta_1 > 0$) and $\frac{\alpha_2}{\alpha_2+\beta_2} < 1$ (resp. $1 < \frac{\alpha_1}{\alpha_1+\beta_1}$), we get $\beta_2 < 0$ (resp. $\beta_1 < 0$). We have $\frac{\alpha_1}{\alpha_2} < 0 < \frac{\beta_1}{\beta_2}$, $\alpha_2 < 0$ and $\beta_2 < 0$, therefore $\alpha_1\beta_2 < \alpha_2\beta_1$.

The last inequality gives $k_1 l_2 < k_2 l_1$, and thus $F_{0,1} \wedge F_{0,2} < 0$. Moreover, from (4.4.7), we have $l_1 < 0$, $l_2 < 0$ and $l_1 - l_2 < 0$. We deduce that the first coordinate of $F_{0,1}$ (resp. $F_{0,2}$, $F_{1,2}$) is positive (resp. negative, negative). Therefore $F_{0,1} = (-l_1, k_1)$, $F_{0,2} = (l_2, -k_2)$ and $F_{1,2} = (l_1 - l_2, k_2 - k_1)$. Recall that $F_{0,3} = (0, -k_3)$, $F_{0,4} = (-l_4, k_4)$ and $F_{3,4} = (l_4, k_3 - k_4)$. We have the following.

- $F_{0,3} \wedge F_{1,2} = k_3 l_4(\beta_1 - \beta_2) = k_3(l_1 - l_2) < 0$, thus $F_{1,2} \notin A_3$.
- $F_{3,4} \wedge F_{0,1} = k_3 l_4(\alpha_1 + \beta_1) = k_1 l_4 - (k_4 - k_3)l_1 > 0$, thus $F_{0,1} \notin A_3$.
- $F_{0,2} \wedge F_{3,4} = k_3 l_4(\alpha_2 + \beta_2) = k_2 l_4 - (k_4 - k_3)l_2 < 0$, thus $F_{0,2} \notin A_3$.

We conclude that the 2-cone A_3 does not contain any 1-cone of $\mathcal{F}_1$, and therefore Δ_1 and Δ_2 do not alternate.

• Assume that $\beta_1 < \beta_2$ and $\tilde{q} < 0 < \tilde{p} < 1$. From the proof of Proposition 4.33, we know that the solutions of $\phi(x) = 1$ are inside $]\tilde{q}, \tilde{p}[$, thus $(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) < 0$ since $\rho_i(x) = \alpha_i - (\alpha_i + \beta_i)x$ for $i = 1, 2$. The fact that $\tilde{p} > 0$ and $\tilde{q} < 0$ implies that $\alpha_1(\alpha_1 + \beta_1) > 0$ and $\alpha_2(\alpha_2 + \beta_2) < 0$. Consequently, we have $\alpha_1\alpha_2 > 0$. Moreover, we have $\alpha_2 < 0$. Indeed, assume on the contrary, that we have $\alpha_2 > 0$. Then $\alpha_1 > 0$, $\alpha_2 + \beta_2 < 0$ and $\alpha_1 + \beta_1 > 0$. Recall that $\frac{\alpha_2}{\alpha_2+\beta_2} < 1$ (resp. $\frac{\alpha_1}{\alpha_1+\beta_1} < 1$), thus $\beta_2 < 0$ (resp. $\beta_1 > 0$), which contradicts $\beta_1 < \beta_2$. Therefore we have $\alpha_1 < 0$, $\alpha_2 + \beta_2 > 0$, $\alpha_1 + \beta_1 < 0$ and thus $\alpha_1 + \beta_1 < \alpha_2 + \beta_2$. From $\alpha_2 + \beta_2 > 0$ (resp. $\alpha_1 + \beta_1 < 0$) and $\frac{\alpha_2}{\alpha_2+\beta_2} < 1$ (resp. $\frac{\alpha_1}{\alpha_1+\beta_1} < 1$), we get $\beta_2 > 0$ (resp. $\beta_1 < 0$). We have $\frac{\beta_1}{\beta_2} < 0 < \frac{\alpha_1}{\alpha_2}$, $\alpha_2 < 0$ and $\beta_2 > 0$, thus $\alpha_1\beta_2 < \alpha_2\beta_1$.

The last inequality gives $k_1 l_2 < k_2 l_1$, and thus $F_{0,1} \wedge F_{0,2} < 0$. Moreover, from (4.4.7), we have $l_1 < 0$ and $0 < l_2$. We deduce that the first coordinate of $F_{0,1}$ (resp. $F_{0,2}$, $F_{1,2}$) is positive (resp. positive, negative), therefore $F_{0,1} = (-l_1, k_1)$, $F_{0,2} = (l_2, -k_2)$ and $F_{1,2} = (l_1 - l_2, k_2 - k_1)$. Therefore we have the following.

- $F_{0,4} \wedge F_{1,2} = k_3 l_4(\alpha_1 - \alpha_2) = k_4(l_2 - l_1) - l_4(k_2 - k_1) > 0$, thus $F_{1,2} \notin A_4$.
- $F_{3,4} \wedge F_{0,1} = k_3 l_4(\alpha_1 + \beta_1) = k_1 l_4 - (k_4 - k_3)l_1 < 0$, thus $F_{0,1} \notin A_4$.
- $F_{0,2} \wedge F_{3,4} = k_3 l_4(\alpha_2 + \beta_2) = k_2 l_4 - (k_4 - k_3)l_2 > 0$, thus $F_{0,2} \notin A_4$.

We conclude that the 2-cone A_4 does not contain any 1-cone of $\mathcal{F}_1$, therefore Δ_1 and Δ_2 do not alternate.

• Assume that $\beta_1 > \beta_2$ and $\tilde{p} < \tilde{q} < 0$. From the proof of Proposition 4.33, we know that the solutions of $\phi(x) = 1$ in I_0 are outside $]\tilde{p}, \tilde{q}[$, thus $(\alpha_1+\beta_1)(\alpha_2+\beta_2) > 0$ since $\rho_i(x) = \alpha_i - (\alpha_i+\beta_i)x$ for $i = 1, 2$. We have that both of $\tilde{q}$ and $\tilde{p}$ are negative, thus $\alpha_2(\alpha_2+\beta_2) < 0$ and $\alpha_1(\alpha_1+\beta_1) < 0$, and consequently we get $\alpha_1\alpha_2 > 0$. Recall that $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, therefore $\alpha_1+\beta_1 > \alpha_2+\beta_2$. Moreover, we have $\frac{1}{\alpha_1+\beta_1} < \frac{1}{\alpha_2+\beta_2}$ since $(\alpha_1+\beta_1)(\alpha_2+\beta_2) > 0$. We have $\beta_1 < 0$. Indeed, assume on the contrary that $\beta_1 > 0$. Then $\alpha_1(\alpha_1+\beta_1) < 0$ gives $\alpha_1 < 0$, and thus $\alpha_2 < 0$. Therefore we get $\alpha_2 < \alpha_1 < 0$ and consequently $0 < \frac{1}{\alpha_1+\beta_1} < \frac{1}{\alpha_2+\beta_2}$ gives $\frac{\alpha_2}{\alpha_2+\beta_2} < \frac{\alpha_1}{\alpha_1+\beta_1}$, which is a contradiction with $\tilde{p} < \tilde{q}$. Then $\beta_2 < \beta_1 < 0$, and $\alpha_1(\alpha_1+\beta_1) < 0$ (resp. $\alpha_2(\alpha_2+\beta_2) < 0$) gives $\alpha_1 > 0$ (resp. $\alpha_2 > 0$) and $(\alpha_1+\beta_1) < 0$ (resp. $(\alpha_2+\beta_2) < 0$). Having $\alpha_2 < \alpha_1$ and $\beta_2 < 0$ (resp. $\alpha_2 > 0$ and $\beta_2 < \beta_1$) gives $\alpha_1\beta_2 < \alpha_2\beta_2$ (resp. $\alpha_2\beta_2 < \alpha_2\beta_1$) and therefore $\alpha_1\beta_2 < \alpha_2\beta_1$.

The last inequality gives $k_1l_2 < k_2l_1$, and thus $F_{0,1} \wedge F_{0,2} < 0$. Moreover, from (4.4.7), we have $l_2 < l_1 < 0$. We deduce that the first coordinate of $F_{0,1}$ (resp. $F_{0,2}$, $F_{1,2}$) is positive (resp. negative, positive), therefore $F_{0,1} = (-l_1, k_1)$, $F_{0,2} = (l_2, -k_2)$ and $F_{1,2} = (l_1 - l_2, k_2 - k_1)$. Therefore we have the following.

- $F_{0,4} \wedge F_{0,2} = k_3l_4\alpha_2 = l_4k_2 - k_4l_2 > 0$, thus $F_{0,2} \notin A_4$.
- $F_{3,4} \wedge F_{0,1} = k_3l_4(\alpha_1+\beta_1) = k_1l_4 - (k_4 - k_3)l_1 < 0$, thus $F_{0,1} \notin A_4$.
- $F_{0,4} \wedge F_{1,2} = k_3l_4(\alpha_1 - \alpha_2) = k_4(l_2 - l_1) - l_4(k_2 - k_1) > 0$, thus $F_{1,2} \notin A_4$.

We conclude that the 2-cone A_4 does not contain any 1-cone of $\mathcal{F}_1$, therefore Δ_1 and Δ_2 do not alternate.

• Assume that $\beta_1 > \beta_2$ and $1 < \tilde{q} < \tilde{p}$. From the proof of Proposition 4.33, we know that the solutions of $\phi(x) = 1$ in I_0 are outside $]\tilde{p}, \tilde{q}[$, thus we have $(\alpha_1+\beta_1)\cdot(\alpha_2+\beta_2) > 0$ since $\rho_i(x) = \alpha_i - (\alpha_i+\beta_i)x$ for $i = 1, 2$. Both of $\tilde{q}$ and $\tilde{p}$ are positive, thus we get $\alpha_2(\alpha_2+\beta_2) > 0$ and $\alpha_1(\alpha_1+\beta_1) > 0$. Consequently, we get that $\alpha_1\alpha_2$ is positive. Recall that $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, therefore $\alpha_1+\beta_1 > \alpha_2+\beta_2$, and thus $\frac{1}{\alpha_1+\beta_1} < \frac{1}{\alpha_2+\beta_2}$ since $(\alpha_1+\beta_1)\cdot(\alpha_2+\beta_2) > 0$. We have $\beta_1 > 0$. Indeed, assume on the contrary, that $\beta_1 < 0$ (and thus $\beta_2 < 0$ since $\beta_2 < \beta_1$). Then $1 < \alpha_1/(\alpha_1+\beta_1)$ (resp. $1 < \alpha_2/(\alpha_2+\beta_2)$) gives $\alpha_1 > 0$ (resp. $\alpha_2 > 0$). Moreover, $\beta_2 < \beta_1 < 0$ (resp. $0 < \frac{\alpha_2}{\alpha_2+\beta_2} < \frac{\alpha_1}{\alpha_1+\beta_1}$) yields $\alpha_1\beta_2 < \alpha_2\beta_1$ (resp. $\alpha_2\beta_1 < \alpha_1\beta_2$), and thus a contradiction. Since $1 < \frac{\alpha_1}{\alpha_1+\beta_1}$ and $\beta_1 > 0$, we get $\alpha_1 < 0$, and thus $\alpha_1+\beta_1 < 0$. Furthermore, this gives $\alpha_2+\beta_2 < 0$ since $(\alpha_1+\beta_1)(\alpha_2+\beta_2) > 0$, and consequently $\alpha_2(\alpha_2+\beta_2) > 0$ yields $\alpha_2 < 0$. We have $\beta_2 > 0$ since $1 < \frac{\alpha_2}{\alpha_2+\beta_2}$, and therefore we get $\alpha_1\beta_2 > \alpha_2\beta_1$ since $0 < \beta_2 < \beta_1$ and $\alpha_2 < \alpha_1 < 0$.

The inequality $\alpha_1\beta_2 > \alpha_2\beta_1$ gives $k_1l_2 > k_2l_1$, and thus $F_{0,1} \wedge F_{0,2} > 0$. Moreover, from (4.4.7), we have $0 < l_2 < l_1$. With these relations we deduce that the first component of $F_{0,1}$ (resp. $F_{0,2}$, $F_{1,2}$) is positive (resp. negative, negative), therefore $F_{0,1} = (l_1, -k_1)$, $F_{0,2} = (-l_2, k_2)$ and $F_{1,2} = (l_2 - l_1, k_1 - k_2)$. Therefore we have the following.

- $F_{0,2} \wedge F_{0,3} = k_3l_4\beta_2 = k_3l_2 > 0$, thus $F_{0,2} \notin A_3$.
- $F_{0,1} \wedge F_{3,4} = k_3l_4(\alpha_1+\beta_1) = k_1l_4 - (k_4 - k_3)l_1 < 0$, thus $F_{0,1} \notin A_3$.
- $F_{3,4} \wedge F_{1,2} = k_3l_4(\alpha_1+\beta_1-\alpha_2-\beta_2) = (k_4 - k_3)(l_2 - l_1) - l_4(k_2 - k_1) > 0$, thus $F_{1,2} \notin A_3$.

We conclude that the 2-cone A_3 does not contain any 1-cone of $\mathcal{F}_1$, therefore Δ_1 and Δ_2 do not alternate.

Chapter 5

Characterization of circuits supporting polynomial systems with the maximal number of positive solutions

Recall that a circuit is a set of $n+2$ points in $\mathbb{R}^n$ that are minimally affinely dependent. In this chapter, we prove the following result.

Theorem 5.1. *A circuit $\mathcal{W}$ in $\mathbb{R}^n$ supports a system with $n+1$ non-degenerate positive solutions if and only if there exists a bijection*

$$\begin{array}{ccc} \{1,\ldots,n+2\} & \longrightarrow & \mathcal{W} \\ i & \longmapsto & w_i \end{array}$$

such that every affine relation on $\mathcal{W}$ can be written as

$$\sum_{i=1}^{s} \alpha_i w_i = \sum_{s+1}^{n+2} \alpha_i w_i,$$

where $s = \lfloor (n+2)/2 \rfloor$ and all α_i, α_i are positive numbers which satisfy

$$\sum_{i=1}^{r} \alpha_i < \sum_{i=s+1}^{s+r} \alpha_i < \sum_{i=1}^{r+1} \alpha_i \quad \text{for} \quad r = 1,\ldots,s-1 \quad \text{if} \quad n \quad \text{is even}$$

or

$$\sum_{i=1}^{r} \alpha_i < \sum_{i=s+2}^{s+r+1} \alpha_i < \sum_{i=1}^{r+1} \alpha_i \quad \text{for} \quad r = 1,\ldots,s-1 \quad \text{if} \quad n \quad \text{is odd.}$$

If Theorem 5.1 is true for any circuit $\mathcal{W} \subset \mathbb{Z}^n$, then it is also true for any circuit $\mathcal{W} \subset \mathbb{R}^n$. Indeed, assume that a system with support a circuit $\mathcal{W} = \{w_1, \ldots, w_{n+2}\} \subset \mathbb{R}^n$ has $n+1$ non-degenerate positive solutions. Then for $i = 1, \ldots, n+2$, points $\tilde{w}_i \in \mathbb{Q}^n$ that are sufficiently close to w_i support a (generalized) polynomial system with the same coefficients and having at least

$n+1$ non-degenerate positive solutions, and thus exactly this number of non-degenerate positive solutions since $n+1$ is an upper bound. Now, multiplying all $\tilde{w}_i$ by some integer, one acquires a system supported on a circuit in $\mathbb{Z}^n$ with $n+1$ non-degenerate positive solutions. Since the inequalities appearing Theorem 5.1 are strict, if the first circuit $\mathcal{W}$ satisfies them, then they are satisfied by the new circuit $\tilde{\mathcal{W}}$ as well, and vice-versa.

Assume that $\mathcal{W} = \{w_1, \ldots, w_{n+2}\}$ is a set of $n+2$ points in $\mathbb{Z}^n$ and consider any affine relation $\sum_{i=1}^{n+2} \lambda_i w_i = 0$ with integer coefficients. After a small perturbation, any system with n equations in n variables $z = (z_1, \ldots, z_n)$ and supported on $\mathcal{W}$ can be reduced by Gaussian elimination to a system

$$z^{w_i} = P_i(z^{w_{n+1}}) \quad \text{for} \quad i = 1, \ldots, n, \tag{5.0.1}$$

having at least the same number of non-degenerate positive solutions, where $P_1, \ldots, P_{n+1}$ are real polynomials of degree 1 in one variable (see Section 5.1). We define in Section 5.1 a real rational function $\varphi(y) = \prod_{i=1}^{n+1} P_i^{\lambda_i}$. We apply *Gale duality* (c.f. [Bih15, BS07, BS08]) to obtain a correspondence between non-degenerate solutions of (5.0.1) and those of $\varphi(y) = 1$. This correspondence restricts to a bijection between non-degenerate positive solutions of the system and the solutions contained in the (possibly empty) interval $\Delta_+ := \{y \in \mathbb{R} |\ P_i(y) > 0 \text{ for } i = 1, \ldots, n+1\}$. After homogenization, we get a real rational map $\mathbb{C}P^1 \to \mathbb{C}P^1$ that we denote again by φ. The *real dessin d'enfant* Γ associated to $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1$, is the inverse image of the real projective line under φ. Given that $\varphi(y) = 1$ has $n+1$ solutions in Δ_+, we deduce by analyzing Γ in Section 5.2 the inequalities of Theorem 5.1. Note that the solutions of $\varphi(y) = 1$ are the roots of

$$G_t(y) = \prod_{\lambda_i > 0} P_i^{\lambda_i}(y) - \prod_{\lambda_i < 0} P_i^{-\lambda_i}(y)$$

in Δ_+.

For the "if" direction of Theorem 5.1, we apply in Section 5.3, Viro patchworking to the polynomial

$$G_t(y) = \prod_{\lambda_i > 0} P_{i,t}^{\lambda_i}(y) - \prod_{\lambda_i < 0} P_{i,t}^{-\lambda_i}(y), \tag{5.0.2}$$

where the $P_{i,t}$ are Viro polynomials of degree 1.

5.1 Technical preamble

Given a system of n polynomials in n variables with total support a circuit $\mathcal{W} = \{w_1, \ldots, w_{n+2}\}$, perturbing slightly its coefficients if necessary, we may assume that the coefficients of $z^{w_1}, \ldots, z^{w_n}$ in the system form an invertible matrix (a small perturbation does not decrease the number of non-degenerate positive solutions). Since we are only interested in non-degenerate positive solutions, we may assume that $w_{n+2} = \mathbf{0}$ and we transform the original via Gaussian elimination into an equivalent system such that the coefficients of $z^{w_1}, \ldots, z^{w_n}$ form a diagonal matrix

$$z^{w_i} = P_i(z^{w_{n+1}}) \quad \text{for} \quad i = 1, \ldots, n, \tag{5.1.1}$$

where $P_i(z^{w_{n+1}}) = a_i + b_i z^{w_{n+1}}$ for $i = 1, \ldots, n$. We start by giving a brief description about *Gale duality* for the system (5.1.1) (c.f. [Bih15, Bih07, BS08]). We use the linear relations on $\mathcal{W}$

to obtain a special polynomial in one variable, called *Gale polynomial.* We have that any integer linear relation among the exponent vectors of $\mathcal{W}$

$$\sum_{i=1}^{n+1} \lambda_i w_i = 0 \tag{5.1.2}$$

gives a monomial identity

$$(z^{w_1})^{\lambda_1} \cdots (z^{w_n})^{\lambda_n} (z^{w_{n+1}})^{\lambda_{n+1}} = 1.$$

If we substitute the polynomials $P_i(z^{w_{n+1}})$ of (5.1.1) into this identity, we obtain a consequence of the latter equation

$$(P_1(z^{w_{n+1}}))^{\lambda_1} \cdots (P_n(z^{w_{n+1}}))^{\lambda_n} (z^{w_{n+1}})^{\lambda_{n+1}} = 1. \tag{5.1.3}$$

Under the substitution $y = z^{w_{n+1}}$, the polynomials $P_i(z^{w_{n+1}})$ become linear functions $P_i(y)$. Set $P_{n+1}(y) = y$. Then (5.1.3) becomes

$$\prod_{i=1}^{n+1} P_i(y)^{\lambda_i} = 1, \tag{5.1.4}$$

which constitutes a **Gale transform** associated to the system (5.1.1). Recall that

$$\Delta_+ = \{y \mid P_i(y) > 0 \text{ for } i = 1, \ldots, n+1\}.$$

We can write equivalently (5.1.4) as $G(y) = 0$, where G is the **Gale polynomial** defined by

$$G(y) = \prod_{\lambda_i > 0} P_i^{\lambda_i}(y) - \prod_{\lambda_i < 0} P_i^{-\lambda_i}(y). \tag{5.1.5}$$

Proposition 5.2. *[BS07] The association*

$$\phi_{w_{n+1}} \ : \ \mathbb{R}_+^n \ni z \ \longmapsto z^{w_{n+1}} =: y \in \mathbb{R}_+$$

is a bijection between solutions $z \in \mathbb{R}_+^n$ of the diagonal system (5.1.1) *and solutions $y \in \Delta_+$ of* (5.1.4) *which restricts to a bijection between their non-degenerate solutions.*

5.2 Proof of the "only if" direction of Theorem 5.1

Set $P_{n+2}(y) = 1$ and $\lambda_{n+2} = -\sum_{i=1}^{n+1} \lambda_i$. We see P_{n+2} as a polynomial of degree 1 having a root at ∞. In what follows, we study the solutions of $\varphi(y) = 1$ contained in Δ_+ where

$$\varphi(y) = \prod_{i=1}^{n+2} P_i^{\lambda_i}(y). \tag{5.2.1}$$

Recall from Chapter 4 that a point $x \in \mathbb{R} \cup \{\infty\}$ is a special point of φ if x is either a root or a pole of φ. Conversely, a non-special critical point $x \in \mathbb{R}$ of φ is a root of φ' such that x is not a special point of φ. In what follows, we see φ (after homogenization) as a real rational map $\mathbb{C}P^1 \to \mathbb{C}P^1$.

Since the graph $\Gamma = \varphi^{-1}(\mathbb{R}P^1)$ is invariant under complex conjugation, it is determined by its intersection with one connected component H (for half) of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$. In all the figures of this chapter, we will only show one half part $H \cap \Gamma$ together with $\mathbb{R}P^1 = \partial H$ represented as a horizontal line. Moreover, for simplicity, we omit the arrows. See Chapter 2 for more details on real dessins d'enfant.

Let a, b be two critical points of φ i.e. vertices of Γ. Recall from Chapter 4 that a and b are neighbours if there is a branch of $\Gamma \setminus \mathbb{R}P^1$ joining them such that this branch does not contain any special or critical points of φ other than a or b. In what follows, we assume that $\varphi(y) = 1$ has $n+1$ solutions contained in Δ_+. Since the latter interval does not contain special points of φ, by Rolle's theorem, the function φ has at least n non-special critical points in Δ_+, and by Remark 5.3, the non-special critical points of φ (all n of them) are contained in Δ_+.

Remark 5.3. *It is proven in [Bih07, proof of Proposition* 2.1*] that*

$$\varphi'(y) = y^{\lambda_{n+1}-1} \prod_{i=1}^{n} P_i^{\lambda_i - 1}(y) \cdot H(y), \tag{5.2.2}$$

where $\deg H \leq n$. *Therefore* φ *has at most* n *non-special critical points.*

Assume that Δ_+ is a non-empty interval. Note that all special points of φ are contained in $\mathbb{R}P^1$, and that by definition, the endpoints of Δ_+ are special points of φ. Choose an orientation of $\mathbb{R}P^1$ and enumerate the special points $x_1, \ldots, x_{n+2}$ of φ with respect to this orientation so that $x_i < x_{i+1}$ for $i = 1, \ldots, n+1$ and the endpoints of Δ_+ are x_1 and x_{n+2} (see Figure 5.1). We also renumber the polynomials P_i so that x_i is the root of P_i for $i = 1, \ldots, n+2$.

Figure 5.1: The domain of positivity Δ_+.

Lemma 5.4. *We have* $\lambda_i \lambda_{i+1} < 0$ *for* $i = 1, \ldots, n+1$.

Proof. Consider a couple x_i, x_{i+1} of two consecutive special points of φ with $i \in \{1, \ldots, n+1\}$. Then these two points are endpoints of an open interval in $\mathbb{R}P^1$ which does not contain special points or non-special critical points. By the cycle rule, this implies that one endpoint is a root (letter p) and the other is a pole (letter q) of φ. □

We will assume that for $i = 1, \ldots, n+2$, we have $\lambda_i > 0$ if i is odd, and $\lambda_i < 0$ if i is even.

Lemma 5.5. *The non-special critical points of* φ *cannot be neighbors to each other.*

Proof. First, note that all special points of φ are contained in $\mathbb{R}P^1 \setminus \Delta_+$. Consider the branch of Γ contained in one of the connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ joining two non-special critical points. Then one of the two connected components of $\mathbb{C}P^1 \setminus \Gamma$ adjacent to this edge will have a boundary disobeying the cycle rule. □

Lemma 5.6. *A special critical point of* φ *cannot be a neighbor to more than one non-special critical point.*

Proof. Assume that there exists a special critical point α of φ that is a neighbor to at least two non-special critical points of φ (in $\mathbb{R}P^1$). Let c_1 and c_2 be two such consecutive non-special critical points. Consider two branches of Γ contained in one of the connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$

joining α to c_1 and α to c_2 respectively. Then one of the two connected components of $\mathbb{C}P^1 \setminus \Gamma$ adjacent to these two branches will have a boundary containing only α as a special point, and thus disobeying the cycle rule. □

Lemma 5.7. *The special points x_1 and x_{n+2} of φ are not neighbors to any of the non-special critical points.*

Proof. Assume that x_1 is a neighbor to a non-special critical point c (the case where x_{n+2} is a neighbor to c is symmetric). Recall that Δ_+ does not contain special points of φ. Consider the branch of Γ contained in one of the connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ joining x_1 to c. Then one of the two connected components of $\mathbb{C}P^1 \setminus \Gamma$ adjacent to this branch will have a boundary containing only x_1 as a special point, and thus disobeying the cycle rule. □

Recall that φ has n non-special critical points all contained in Δ_+. Let $c_2, \ldots, c_{n+1}$ denote these points numbered so that $x_{n+2} < c_{n+1} < c_n < \cdots < c_2 < x_1$.

Proposition 5.8. *For $i = 2, \ldots, n+1$, the special point x_i is a neighbor to c_i (see Figure 5.2).*

Proof. First, by Lemma 5.7, we have that the roots of P_1 and P_{n+2} are not neighbors to non-special critical points. Recall that there exists n non-special critical points in Δ_+. Therefore, by Lemmata 5.5 and 5.6, we have that for $i = 2, \cdots, n+1$, the special point x_i is a neighbor to only one non-special critical point c_j. Consider the closed interval $I \subset \mathbb{R}P^1$ with endpoints x_i and c_j and which contains x_1. The special points in I are $x_1, x_2, \ldots, x_i$ and the non-special critical points in I are $c_2, \ldots, c_j$. Then the non-special critical points in I can only be neighbors to special points in $I \setminus \{x_1\}$ (see Lemma 5.7). This induces a bijection between $\{x_2, \ldots, x_i\}$ and $\{c_2, \ldots, c_j\}$, thus $i = j$.

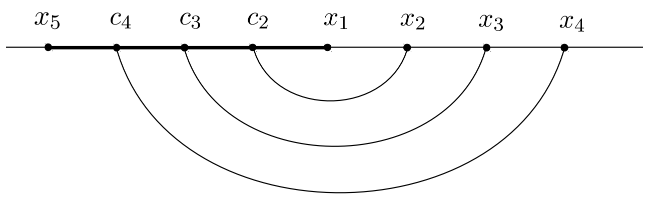

Figure 5.2: The graph Γ satisfying Proposition 5.8 for $n = 3$.

□

Lemma 5.9. *The special point x_1 (resp. x_{n+2}) of φ can only be a neighbor to the special point x_2 (resp. x_{n+1}) of φ.*

Proof. We prove the result only for x_1 since the case for x_{n+2} is symmetric. Consider the open interval I with endpoints c_2 and x_2 containing x_1. By Proposition 5.8, we have that c_2 and x_2 are neighbors. The result comes as a consequence of Lemma 5.7 and of the fact that there does not exist special points or non-special critical points in I other than x_1 (See Figure 5.3). □

Lemma 5.10. *For $i = 1, \ldots, n$, the only special points which can be neighbors to x_{i+1} are x_i and x_{i+2}.*

Proof. Assume first that $i = 1$ (the case $i = n$ is symmetric). Recall that by Proposition 5.8, the special point x_2 (resp. x_3) and c_2 (resp. c_3) are neighbors. Therefore, the only other possible neighbors to x_2 are x_1 and x_3 (see Figure 5.3).

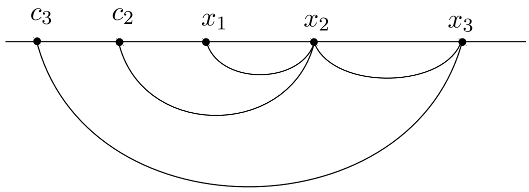

Figure 5.3: The special point x_2 can only be neighbours to x_1 or x_3.

Assume now that $i \neq 1$ and $i \neq n$. Recall that by Proposition 5.8 the point x_i (resp. x_{i+2}) is a neighbor to c_i (resp. c_{i+2}). Consider the open disc $\mathcal{C}$ in $\mathbb{C}P^1$ with boundary given by the union of $[c_{i+2}, c_i]$, $[x_i, x_{i+2}]$ and the complex arcs of Γ joining c_i to x_i (resp. c_{i+2} to x_{i+2}), and which are contained in one given connected component of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ (see Figure 5.4). The result follows from the fact that the only special points in the boundary of $\mathcal{C}$ are x_i, x_{i+1} and x_{i+2}.

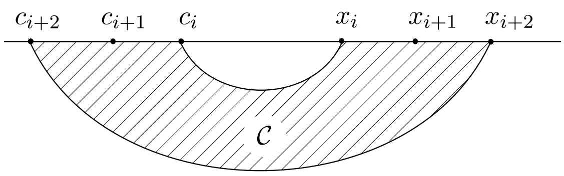

Figure 5.4: The region $\mathcal{C} \subset \mathbb{C}P^1 \setminus \Gamma$ together with its boundary.

□

Recall that λ_i is positive if i is odd and negative if i is even, and thus the root x_i of P_i is a zero (resp. pole) of φ if i is odd (resp. even). Recall that the valency of any special point x_i is the number V_i of edges of Γ that are incident to x_i.

For $i = 1, \dots, n+1$, denote by $N_{i,i+1}$ the number of edges of Γ in $\mathbb{C}P^1 \setminus \mathbb{R}P^1$ joining the special points x_i and x_{i+1}. By Lemmata 5.7 and 5.9, we have $V_1 = N_{1,2} + 2$ and $V_{n+2} = N_{n+1,n+2} + 2$ (each number 2 corresponds to the pair of edges of Γ in $\mathbb{R}P^1$ incident to x_1 and x_{n+2} respectively). Moreover, for $i = 2, \dots, n+1$, Proposition 5.8 and Lemma 5.10 show that $V_i = N_{i-1,i} + N_{i,i+1} + 4$, where the number 4 counts the branches in $\mathbb{R}P^1$ together with the branches joining x_i to c_i. Knowing that $V_i = |2\lambda_i|$, it is straightforward to compute that for $k = 1, \dots, \lfloor n/2 \rfloor + 1$, we have

$$\sum_{j=1}^{k} \lambda_{2j-1} < -\sum_{j=1}^{k} \lambda_{2j} < \sum_{j=0}^{k} \lambda_{2j+1} \quad \text{if } n \text{ is even, or} \tag{5.2.3}$$

$$\sum_{j=1}^{k} \lambda_{2j-1} < -\sum_{j=1}^{k} \lambda_{2j} < \sum_{j=0}^{k} \lambda_{2j+1} \quad \text{if } n \text{ is odd.} \tag{5.2.4}$$

This finishes the proof of the "only if part" of Theorem 5.1.

We now finish the description of Γ. For $i \in \{0, \ldots, n+1\}$, consider the real branch L_0 joining two consecutive special points x_i and x_{i+1} of φ. Let $k := N_{i,i+1}/2$, and for $j = 1, \ldots, k$, consider the couple of conjugate branches $(L_j, \overline{L}_j)$ joining x_i to x_{i+1} enumerated such that the open disc of $\mathbb{C}P^1$ with boundary $(L_j, \overline{L}_j)$ and containing L_0, contains the couple $(L_{j-1}, \overline{L}_{j-1})$ as well (assuming that $L_0 \equiv \overline{L}_0$). The branch L_k (resp. $\overline{L}_k$) does not contain a letter r since there exists a cycle of Γ_1 containing both L_k (resp. $\overline{L}_k$) and a letter $r \in \Delta_+$, and thus obeying the cycle rule. On the other hand, the branch L_{k-1} (resp. $\overline{L}_{k-1}$) contains a letter r where the cycle formed by the union of L_k and L_{k-1} (resp. $\overline{L}_k$ and $\overline{L}_{k-1}$) and containing x_i and x_{i+1} obeys the cycle rule. We deduce that for $j = 0, \ldots, k$, the branch L_j (resp. $\overline{L}_j$) has exactly 1 or 0 letters r according as j and $k-1$ have the same parity or not (see Example 5.12).

In fact, this complete description of the dessin d'enfant Γ can be used to prove the "if" part of Theorem 5.1 with the same techniques as in [Bih07]. However, we choose in Section 5.3 a different method, namely Viro's combinatorial patchworking, which shows clearly why the inequalities of Theorem 5.1 are necessary.

Remark 5.11. *From the relations described above, we see that the collection of integers $N_{i,i+1}$ is determined by the collection of the coefficients λ_i (and vice-versa). Moreover, we see that the inequalities of Theorem 5.1 are equivalent to $N_{i,i+1} \geq 0$ for $i = 1, \ldots, n+1$.*

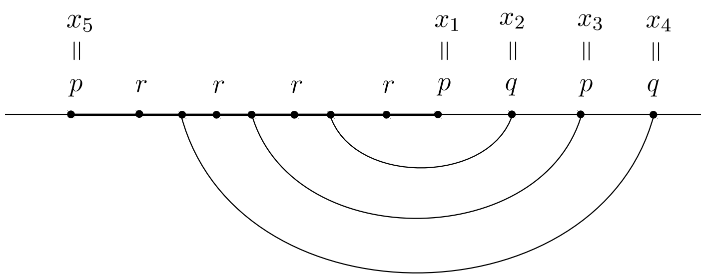

Figure 5.5: The dessin d'enfant Γ_0 for $n = 3$.

Example 5.12. *Figure 5.6 represents an example of Γ where $n = 3$, $\lambda_1 = 3$, $\lambda_2 = -7$, $\lambda_3 = 6$, $\lambda_4 = -3$ and $\lambda_5 = 1$. The dessin d'enfant Γ can be obtained from Γ_0 (see Figure 5.5) by adding complex branches connecting consecutive special points and letters r as described above.*

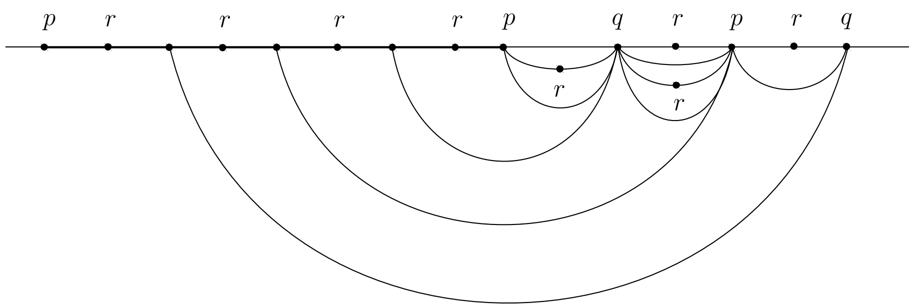

Figure 5.6: An example of a dessin d'enfant Γ for $n = 3$ that can be constructed from Γ_0.

5.3 Proof of the "if" direction of Theorem 5.1

Assume that $\lambda_i > 0$ if i is odd, $\lambda_i < 0$ if i is even and (5.2.3) or (5.2.4) is satisfied (depending on the parity of n). In this section, we construct polynomials P_i (see Section 5.2) such that (5.2.1) has $n+1$ solutions in Δ_+. These polynomials have the form $P_{1,t}(y) = t^{\alpha_1}y$, $P_{n+2,t}(y) = 1$ and $P_{i,t}(y) = 1 + t^{\alpha_i}y$ for $i = 2, \dots, n+1$, where t is a real positive parameter that will be taken small enough, and each α_i is a real number. The corresponding Gale polynomial (5.1.5) is

$$G_t(y) := \prod_{j=0}^{\lfloor n/2 \rfloor} P_{2j+1,t}^{\lambda_{2j+1}}(y) - \prod_{j=1}^{\lfloor (n+1)/2 \rfloor} P_{2j,t}^{-\lambda_{2j}}(y). \tag{5.3.1}$$

We are interested in the roots of G_t contained in $\Delta_{+,t}$, which is the common positivity domain of the polynomials $P_{i,t}$. Note that here $\Delta_{+,t} =]0, +\infty[$. The polynomial G_t is a particular case of a Viro polynomial (c.f. [BBS06, Bih02, Vir84])

$$f_t(y) = \sum_{p=p_0}^{d} \phi_p(t) y^p,$$

where t is a positive real number, and each coefficient $\phi_p(t)$ is a finite sum $\sum_{q \in I_p} c_{p,q} t^q$ with $c_{p,q} \in \mathbb{R}$ and q a real number.

We now recall how one can recover in some cases the real roots of f_t for t small enough (see for instance [BBS06]). Write f for the function of y and t defined by f_t. Let $D \subset \mathbb{R}^2$ be the convex hull of the points (p, q) for $p_0 \le p \le d$ and $q \in I_p$. Assume that D has dimension 2. Its lower hull L is the union of the edges $e_1, \dots, e_l$ of D whose inner normals have positive second coordinate. Let I_i be the image of e_i under the projection $\mathbb{R}^2 \to \mathbb{R}$ forgetting the last coordinate. Then the intervals $I_1, \dots, I_l$ subdivide the Newton segment $[p_0, d]$ of f_t. Let $f^{(i)}$ be the facial subpolynomial of f for the face e_i. That is, the polynomial $f^{(i)}$ is the sum of terms $c_{p,q}y^p$ such that $(p, q) \in e_i$. Suppose that e_i is the graph of $y \mapsto a_i y + b_i$ over I_i. Expanding $f_t(yt^{-a_i})/t^{b_i}$ in powers of t gives

$$f_t(yt^{-a_i})/t^{b_i} = f^{(i)}(y) + g^{(i)}(y, t) \quad \text{and} \quad i = 1, \dots, l, \tag{5.3.2}$$

where $g^{(i)}(y, t)$ collects the terms whose powers of t are positive. Then $f^{(i)}(y)$ has Newton segment I_i and its number of non-zero roots counted with multiplicities is $|I_i|$, the length of the interval I_i.

Lemma 5.13. *Assume that for $i = 1, \dots, l$, the polynomial $f^{(i)}$ is a binomial. Then there exists a bijection between the set of all non-degenerate positive roots of f_t for $t > 0$ small enough and the set of non-degenerate positive roots of $f^{(1)}, \dots, f^{(l)}$.*

Proof. Since $f^{(i)}(y)$ is a binomial, it has at most one positive root r which is simple, and there will be a positive root $r_{i,t}$ of

$$f^{(i)}(y) + g^{(i)}(y, t)$$

near such r for t small enough. Let $K \subset]0, +\infty[$ denote a compact interval containing the positive root of $f^{(i)}$ for $i = 1, \dots, l$. Then, for $t > 0$ small enough, the interval K contains the positive root $r_{i,t}$ of $f_t(yt^{-a_i})/t^{b_i}$. Moreover, the intervals $t^{-a_1}K, \dots, t^{-a_l}K$ are disjoint for $t > 0$ small enough. This gives l positive roots of f_t for $t > 0$ small enough. Roots of $f_t(yt^{-a_i})/t^{b_i}$ which are close to a point r are positive only if r is positive, and the number of these roots is determined by the first term $f^{(i)}(y)$. Since $f^{(i)}(y)$ is a binomial, it has only one simple positive root. □

To simplify the notations, set $p_0 = 0$, $p_1 = \lambda_1$, $p_2 = -\lambda_2$, $p_3 = \lambda_1 + \lambda_3, \dots$ and $p_{n+1} = \sum_{j=0}^{n/2} \lambda_{2j+1}$ if n is even and $p_{n+1} = -\sum_{j=1}^{(n+1)/2} \lambda_{2j}$ if n is odd. Then by assumption, we have $p_0 < p_1 < \cdots < p_{n+1}$. Set $h_0 = 0$ and choose real numbers $h_1, \dots, h_{n+1}$ such that the lower part L of the convex hull of $\{(p_i, h_i) |\ i = 0, \dots, n+1\}$ consists of the segments $[(p_i, h_i), (p_{i+1}, h_{i+1})]$ for $i = 0, \dots, n$. Therefore, projecting L to $\mathbb{R}$ via the map $\mathbb{R}^2 \to \mathbb{R}$ forgetting the last coordinate, we get the subdivision of $[0, p_{n+1}]$ by the intervals $[p_i, p_{i+1}]$ (see Figure 5.7). Set $\alpha_1 = h_1/p_1$, $\alpha_2 = h_2/p_2$ and

$$\alpha_i = \frac{h_i - h_{i-2}}{p_i - p_{i-2}} \quad \text{for} \quad i = 3, \dots, n+1.$$

Proposition 5.14. *For $t > 0$ small enough the polynomial* (5.3.1) *has $n+1$ roots in $\Delta_{+,t} =]0, +\infty[$.*

Proof. It is easy to see that the lower hull of the Viro polynomial

$$\prod_{j=0}^{\lfloor n/2 \rfloor} P_{2j+1,t}^{\lambda_{2j+1}}(y) \tag{5.3.3}$$

is composed of the segments $[(p_{2j+1}, h_{2j+1}), (p_{2j+3}, h_{2j+3})]$ for $j = 0, \dots, \lfloor n/2 \rfloor - 1$. Similarly, the lower hull of

$$-\prod_{j=1}^{\lfloor (n+1)/2 \rfloor} P_{2j,t}^{-\lambda_{2j}}(y) \tag{5.3.4}$$

is composed of the segments $[(p_{2j-2}, h_{2j-2}), (p_{2j}, h_{2j})]$ for $j = 1, \dots, \lfloor (n+1)/2 \rfloor$. It follows that the lower hull of the Viro polynomial G_t is L. Now we apply Lemma 5.13 to G_t. For $i = 0, \dots, n$, the facial subpolynomial $G^{(i)}$ corresponding to the segment $[(p_i, h_i), (p_{i+1}, h_{i+1})] \subset L$ is a binomial where one monomial comes from (5.3.3) and the other comes from (5.3.4). Consequently, this binomial has coefficients of different signs and thus it has one simple positive root. Therefore by Lemma 5.13, the polynomial G_t has $n+1$ non-degenerate positive roots for $t > 0$ small enough. □

Example 5.15. *Choose for $i = 0, \dots, n$, the slope of the segment $[(p_i, h_i), (p_{i+1}, h_{i+1})]$ of L to be equal to i. We compute explicitly the values $\alpha_1, \dots, \alpha_{n+1}$ of the exponent of t appearing respectively in $P_{1,t}, \dots, P_{n+1,t}$. We have $h_1 = 0$, and*

$$i = \frac{h_{i+1} - h_i}{p_{i+1} - p_i} \quad \textit{for } i = 0, \dots, n.$$

Since $\alpha_1 = 0$ and for $i = 0, \dots, n-1$, we have $\alpha_{i+2} = (h_{i+2} - h_i)/(p_{i+2} - p_i)$, then

$$\alpha_{i+2} = i + \frac{p_{i+2} - p_{i+1}}{p_{i+2} - p_i}.$$

Note that $p_{i+2} - p_i = \lambda_i$ if i is odd, and $p_{i+2} - p_i = -\lambda_i$ if i is even. Moreover, we have

$$p_{i+2} - p_{i+1} = \sum_{j=0}^{(i+1)/2} \lambda_{2j+1} + \sum_{j=1}^{(i+1)/2} \lambda_{2j} \quad \textit{if } i \textit{ is odd and} \quad -\sum_{j=1}^{(i+2)/2} \lambda_{2j} - \sum_{j=0}^{i/2} \lambda_{2j+1} \quad \textit{if } i \textit{ is even.}$$

Therefore,

$$\alpha_{i+2} = i + \frac{\sum_{j=0}^{\lfloor i+1 \rfloor/2} \lambda_{2j+1} + \sum_{j=1}^{\lfloor i+2 \rfloor/2} \lambda_{2j}}{\lambda_i}.$$

□

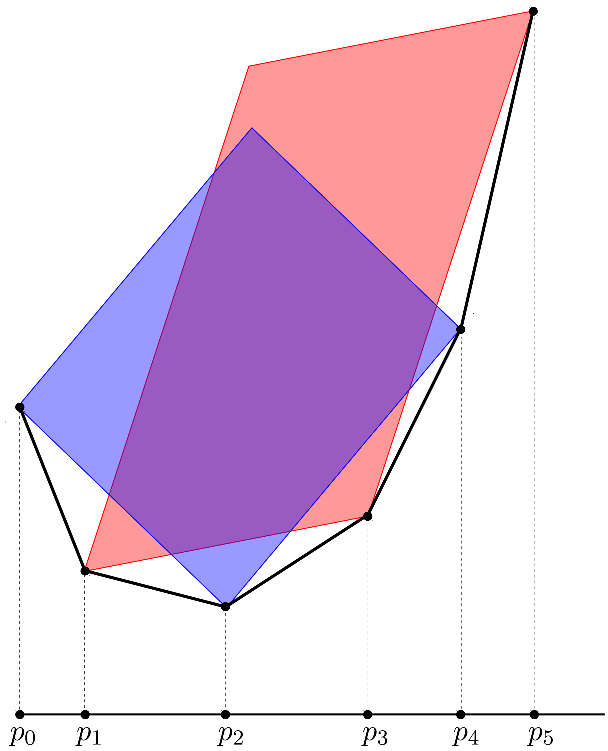

Figure 5.7: The lower hull L of G_t for $n = 4$.

Chapter 6

Constructing polynomial systems with many positive solutions

6.1 Statement of the main results

Consider a system defined on the field of real generalized locally convergent Puiseux series with two equations in two variables supported on a set of five distinct points in $\mathbb{Z}^2$. We say that such system is of type $n = k = 2$. Moreover, we assume that no three points of the support belong to a line, and we say that such a system is *highly non-degenerate*.

6.1.1 For normalized systems

Given such a system, we prove in Section 6.3 that one can associate to it a system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0, \end{aligned} \tag{6.1.1}$$

with equations in $\mathbb{RK}[y_1^{\pm 1}, y_2^{\pm 1}]$, that has the same number of positive non-degenerate solutions, and satisfying that all a_i and b_j belong to $\mathbb{RK}^*$ and verify $\text{ord}(a_i) = \text{ord}(b_j) = 0$, all m_i, n_i belong to $\mathbb{Z}$ with $m_1, n_2 > 0$, and both α, β are real numbers. A highly non-degenerate system 6.1.1 satisfying the latter conditions is called a ***normalized system***.

We prove in Section 6.5 the following result.

Theorem 6.1. *If* $(\alpha, \beta) \neq (0, 0)$*, then* (6.1.1) *has at most nine non-degenerate positive solutions.*

In Subsection 6.5.2, we construct a system (6.1.1) having seven non-degenerate positive solutions, and thus proving the following.

Theorem 6.2. *There exists a system* (6.1.1) *having seven non-degenerate positive solutions.*

In the last two sections of this chapter, we refine Theorem 6.1 by proving the following result.

Theorem 6.3. *If* $\alpha \neq \beta$ *or* $\alpha = \beta < 0$*, then the sharp bound on the number of positive solutions of* (6.1.1) *is six.*

We prove in Section 6.6 Theorem 6.3 when $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ for $i = 0, 2$, and in Section 6.7, we prove this result when

$$\alpha\beta \neq 0, \quad \frac{\mathrm{coef}(a_0)}{\mathrm{coef}(b_0)} \neq \frac{\mathrm{coef}(a_2)}{\mathrm{coef}(b_2)} \quad \text{and} \quad \mathrm{coef}(a_i) \neq \mathrm{coef}(b_i) \quad \text{for} \quad i = 0, 2.$$

In fact, due to Lemmata 6.30 and 6.31 of Section 6.4, the conditions of Sections 6.6 and 6.7 are complementary given that $(\alpha, \beta) \neq (0, 0)$.

Theorem 6.3 was merely to give a direction to follow in order to construct a system (6.1.1) that has more than six non-degenerate positive solutions.

6.1.2 Transversal intersection points

Consider a (not necessarily normalized) system

$$f_1 = f_2 = 0 \tag{6.1.2}$$

of type $n = k = 2$, where $f_1, f_2 \in \mathbb{R}\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]$. Assume that the tropical curves T_1 and T_2 associated to f_1 and f_2 intersect transversally. Let $\mathcal{W}_1, \mathcal{W}_2 \subset \mathbb{Z}^2$ denote the supports of f_1 and f_2 respectively. Note that $|\mathcal{W}_1 \cup \mathcal{W}_2| = 5$. Then by [Bih14, Theorem 1.1], the following result implies that the number of intersection points of T_1 and T_2 is at most six.

Lemma 6.4. *The discrete mixed volume (see* (2.2.3) *in Subsection 2.2.5 of Chapter 2)* $D(\mathcal{W}_1, \mathcal{W}_2)$ *does not exceed six.*

Proof. We distinguish the five possible cases $|\mathcal{W}_1 \cap \mathcal{W}_2| = i$ for $i = 1, \ldots, 5$, and prove the result for $i = 3, 4$ since the case $i = 5$ is proven in [Bih14] and the other cases are similar. The discrete mixed volume of $\mathcal{W}_1$ and $\mathcal{W}_2$ is expressed as

$$D(\mathcal{W}_1, \mathcal{W}_2) = |\mathcal{W}_1 + \mathcal{W}_2| - |\mathcal{W}_1| - |\mathcal{W}_2| + 1. \tag{6.1.3}$$

Assume first that $|\mathcal{W}_1 \cap \mathcal{W}_2| = 4$. Then the cardinal of one of the two sets, say $\mathcal{W}_1$, is equal to four. Writing $\mathcal{W}_1 = \{w_0, w_1, w_2, w_3\}$ and $\mathcal{W}_2 = \{w_0, w_1, w_2, w_3, w_4\}$, we get

$$\mathcal{W}_1 + \mathcal{W}_2 = \bigcup_{i=0}^{3} \{w_i + w_j \mid j = 0, \ldots, 4, \ j \geq i\},$$

and thus $|\mathcal{W}_1 + \mathcal{W}_2| \leq 14$. Therefore, with $|\mathcal{W}_1| = 4$ and $|\mathcal{W}_2| = 5$, we deduce that $D(\mathcal{W}_1, \mathcal{W}_2) \leq 6$.

Assume now that $|\mathcal{W}_1 \cap \mathcal{W}_2| = 3$. We distinguish two cases

i) First case: $|\mathcal{W}_1| = 3$ and $|\mathcal{W}_2| = 5$ (the case where $|\mathcal{W}_1| = 5$ and $|\mathcal{W}_2| = 3$ is symmetric). Writing $\mathcal{W}_1 = \{w_0, w_1, w_2\}$ and $\mathcal{W}_2 = \{w_0, w_1, w_2, w_3, w_4\}$, we get

$$\mathcal{W}_1 + \mathcal{W}_2 = \bigcup_{i=0}^{2} \{w_i + w_j \mid j = 0, \ldots, 4, \ j \geq i\},$$

and thus $|\mathcal{W}_1 + \mathcal{W}_2| \leq 12$. Therefore, with $|\mathcal{W}_1| = 3$ and $|\mathcal{W}_2| = 5$, we deduce that $D(\mathcal{W}_1, \mathcal{W}_2) \leq 5$.

ii) Second case: $|\mathcal{W}_1| = |\mathcal{W}_2| = 4$. Writing $\mathcal{W}_1 = \{w_0, w_1, w_2, w_3\}$ and $\mathcal{W}_2 = \{w_1, w_2, w_3, w_4\}$, we get

$$\mathcal{W}_1 + \mathcal{W}_2 = \bigcup_{i=0}^{3} \{w_i + w_j \mid j = 1, \ldots, 4,\ j \geq i\},$$

and thus $|\mathcal{W}_1 + \mathcal{W}_2| \leq 13$. Therefore, with $|\mathcal{W}_1| = 4$ and $|\mathcal{W}_2| = 4$, we deduce that $D(\mathcal{W}_1, \mathcal{W}_2) \leq 6$.

□

We prove that the bound in Lemma 6.4 is sharp and in fact can be realized by *positive* intersection points of two tropical curves.

Proposition 6.5. *There exist two plane tropical curves T_1 and T_2 defined by equations containing a total of five monomials and which have six positive transversal intersection points.*

An explicit system proving Proposition 6.5 is given in Example 6.38 (see Subsection 6.4.1).

6.2 Non-transversal intersection components of type (I)

Consider the polynomials

$$f(y) := \sum_{i=0}^{r} \mu_i y^{v_i} \quad \text{and} \quad g(y) := \sum_{i=0}^{s} \nu_i y^{w_i},$$

where f and g belong to $\mathbb{R}\mathbb{K}[y_1^{\pm 1}, y_2^{\pm 1}]$. Let Δ_f and Δ_g (resp. τ_f and τ_g, T_f and T_g) denote the Newton polytopes (resp. dual subdivisions, tropical curves) associated to f and g respectively. Consider the system

$$f = g = 0, \tag{6.2.1}$$

with total support not contained in any hyperplane of $\mathbb{R}^2$ and satisfying that all solutions of (6.2.1) in $(\mathbb{K}^*)^2$ are non-degenerate.

If ξ is an isolated point of $T_f \cap T_g$, we have that $z \mapsto \operatorname{coef}(z)$ induces a bijection from the set of non-degenerate solutions in $(\mathbb{R}\mathbb{K}_{>0})^2$ of the system (6.2.1) with valuation ξ to the set of non-degenerate positive solutions of the reduced system with respect to ξ (see Proposition 2.23). When ξ is not a point (i.e. an intersection of type (I)), some of the points in the relative interior $\mathring{\xi}$ of ξ are not valuations of solutions of (6.2.1) in $(\mathbb{K}^*)^2$. In fact, we are interested in positive solutions of (6.2.1). Here, we give a way to compute

$$\operatorname{Val}\left(\{z \in (\mathbb{R}\mathbb{K}_{>0})^2 \mid f(z) = g(z) = 0\}\right) \cap \mathring{\xi}$$

and the coefficients of the first order terms of $\{z \in (\mathbb{R}\mathbb{K}_{>0})^2 \mid f(z) = g(z) = 0\}$ with valuation in $\mathring{\xi}$ (see Remark 6.7).

Assume that T_f and T_g have a non-transversal intersection component ξ of type (I) and that $\mathring{\xi}$ contains the valuations of positive solutions of the latter system. Recall that $\mathring{\xi}$ is the relative interior of the intersection of a face ξ_f of T_f and a face ξ_g of T_g satisfying $\dim(\xi_f) = \dim(\xi_g) = \dim(\xi_f \cap \xi_g) = 1$. Assume that each of $\Delta_{\xi_f} \cap \mathbb{Z}^2$ and $\Delta_{\xi_g} \cap \mathbb{Z}^2$ has only two points belonging to the support of f and g respectively so that these points are endpoints of Δ_{ξ_f} and Δ_{ξ_g} respectively. In this section, we introduce a method for computing the valuations in $\mathring{\xi}$ of non-degenerate positive solutions of (6.2.1).

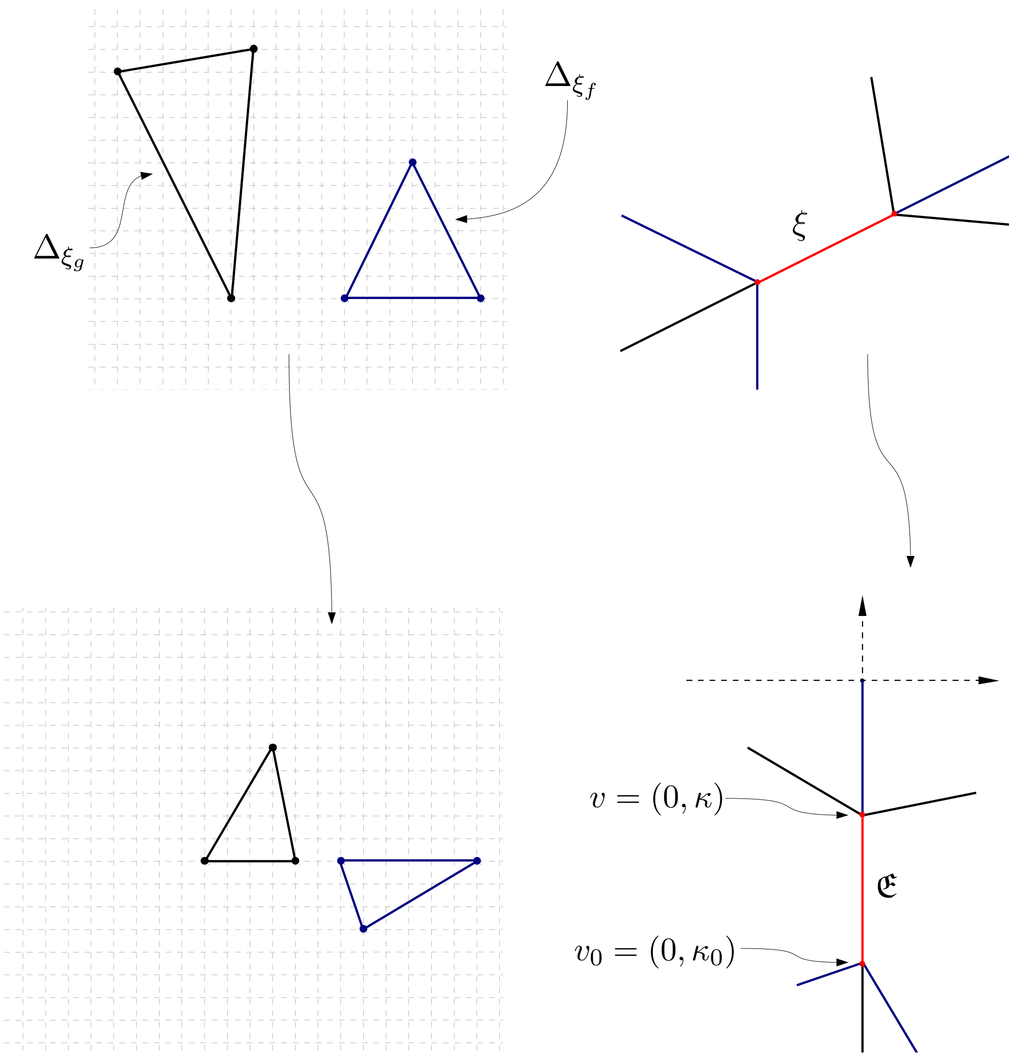

Figure 6.1: A monomial change of coordinates that acts on the type-(I) intersection cell.

Proposition 6.6. *There exists a system*

$$c_0 + c_1 y_1^{k_1} + \sum_{i=2}^{r} c_i y_1^{k_i} y_2^{l_i} = d_0 + d_1 y_1^{m_1} + \sum_{i=2}^{s} d_i y_1^{m_i} y_2^{n_i} = 0 \qquad (6.2.2)$$

defined by polynomials in $\mathbb{RK}[y_1^{\pm 1}, y_2^{\pm 1}]$ *which satisfies the following properties.*

i) $\mathrm{coef}(c_0) = \mathrm{coef}(d_0) = -1$, $\mathrm{coef}(c_1) = \mathrm{coef}(d_1) = 1$, $\mathrm{ord}(c_0) = \mathrm{ord}(d_0) = \mathrm{ord}(c_1) = \mathrm{ord}(d_1) = 0$ *and* k_1, m_1 *are positive integers. The tropical curves associated to* (6.2.2) *intersect non-transversally at a cell* $\mathfrak{E}$ *of type (I) contained in* $\{0\}\times\,]-\infty, 0[$ *with endpoints*

$v = (0,\kappa)$ and $v_0 = (0,\kappa_0)$, where

$$\kappa = \max\{x_2 \mid 0 = \max\{\mathrm{val}(c_i) + l_i x_2\ ,\ \mathrm{val}(d_i) + n_i x_2 \mid i = 2,\dots,r\ ,\ i = 2,\dots,s\}\}$$

and

$$\kappa_0 = \min\{x_2 \mid 0 = \max\{\mathrm{val}(c_i) + l_i x_2\ ,\ \mathrm{val}(d_i) + n_i x_2 \mid i = 2,\dots,r\ ,\ i = 2,\dots,s\}\}$$

(we may have $\kappa_0 = -\infty$ when $\mathfrak{E}$ is unbounded).

ii) *The systems (6.2.1) and (6.2.2) have the same number of non-degenerate solutions in $(\mathbb{K}^*)^2$. Moreover, they have the same number of non-degenerate positive solutions with valuations in $\mathring{\xi}$ and $\mathring{\mathfrak{E}}$ respectively.*

Proof. In what follows, we make transformations on (6.2.1) to obtain the system (6.2.2) so that (6.2.1) and (6.2.2) have the same number of non-degenerate solutions in $(\mathbb{K}^*)^2$. Moreover, the latter transformation maps each non-degenerate positive solution of (6.2.1) with valuation in $\mathring{\xi}$ to a non-degenerate positive solution of (6.2.2) with valuations in $\mathring{\mathfrak{E}}$ so that this mapping is a bijection. The intersection component ξ has a direction orthogonal to the edge $\Delta_{\xi_f} \in \tau_f$ dual to ξ_f and to the edge $\Delta_{\xi_g} \in \tau_g$ dual to ξ_g, thus both these segments are parallel. Enumerate the exponent vectors $v_0,\dots,v_r$ and $w_0,\dots,w_s$ so that the equations defining the relative interiors of ξ_f and ξ_g are expressed as

$$\{x \in \mathbb{R}^2 |\ \langle x, v_0\rangle + \mathrm{val}(\mu_0) = \langle x, v_1\rangle + \mathrm{val}(\mu_1) > \max_{i=2}^{r}(\langle x, v_i\rangle + \mathrm{val}(\mu_i))\}$$

and

$$\{x \in \mathbb{R}^2 |\ \langle x, w_0\rangle + \mathrm{val}(\nu_0) = \langle x, w_1\rangle + \mathrm{val}(\nu_1) > \max_{i=2}^{s}(\langle x, w_i\rangle + \mathrm{val}(\nu_i))\}$$

respectively, and so that $\lambda(v_1 - v_0) = (w_1 - w_0)$ for some $\lambda \in \mathbb{R}^*_+$. The endpoints of Δ_{ξ_f} and Δ_{ξ_g} are v_0, v_1 and w_0, w_1 respectively. Moreover, one can assume that $v_0 = w_0 = (0,0)$. Doing a monomial change of coordinates if necessary, we may assume that both these edges are horizontal (zero second coordinate), and $v_1 = (0,k_1)$ and $w_1 = (0,m_1)$ for some positive integers k_1 and m_1. Set $\mathrm{coef}(\mu_0) = \mathrm{coef}(\nu_0) = -1$ by dividing the first (resp. second) equation of (6.2.1) by $-\mathrm{coef}(\mu_0)$ (resp. $-\mathrm{coef}(\nu_0)$). Since $\mathring{\xi}$ contains valuations of positive solutions of (6.2.1), the reduced system

$$-1 + \mathrm{coef}(\mu_1) y_1^{k_1} = -1 + \mathrm{coef}(\nu_1) y_1^{m_1} = 0 \tag{6.2.3}$$

has a positive solution

$$y_1 = \left(\frac{1}{\mathrm{coef}(\mu_1)}\right)^{\frac{1}{k_1}} = \left(\frac{1}{\mathrm{coef}(\nu_1)}\right)^{\frac{1}{m_1}}.$$

Set $\mathrm{coef}(\mu_1) = \mathrm{coef}(\nu_1) = 1$ by replacing y_1 by $(1/\mathrm{coef}(\mu_1))^{(1/k_1)} y_1$ in (6.2.1). Without loss of generality, we may assume that $\mathrm{ord}(\mu_0) = \mathrm{ord}(\nu_0) = 0$. Denote $v_i = (k_i, l_i)$ and $w_i = (m_i, n_i)$ for $i = 2,\dots,r$ and $i = 1,\dots,s$. Since $v_0 = w_0 = (0,0)$, a point $(x_1,x_2) \in \mathbb{R}^2$ belonging to ξ satisfies $0 = k_1 x_1 + \mathrm{val}(\mu_1) > \max\{k_i x_1 + l_i x_2 + \mathrm{val}(\mu_i), i = 2,\dots,r\}$ and $0 = m_1 x_1 + \mathrm{val}(\nu_1) > \max\{m_i x_1 + n_i x_2 + \mathrm{val}(\nu_i),\ i = 2,\dots,s\}$, and thus $\mathrm{val}(\mu_1)/k_1 = \mathrm{val}(\nu_1)/m_1$. Set $\mathrm{val}(\mu_1) = \mathrm{val}(\nu_1) = 0$ by replacing y_1 by $t^{\mathrm{val}(\mu_1)/k_1} y_1$ in (6.2.1). The cell ξ is now contained in the second-coordinate axis of $\mathbb{R}^2$. Recall that ξ is either a segment or of a half-line. Replacing y_2

by $t^\gamma y_2$ in (6.2.1) for some real number γ translates $T_f \cup T_g$ vertically, and y_2 by y_2^{-1} acts as a symmetry on $T_f \cup T_g$ with respect to the first-coordinate axis of $\mathbb{R}^2$. We use these transformations so that the resulting $\overset{\circ}{\xi}$ is situated entirely below the first-coordinate axis of $\mathbb{R}^2$. Therefore, an endpoint v of ξ is a point $(0, x_2) \in \mathbb{R}^2$ satisfying $0 = k_1 x_1 \geq \max\{\mathrm{val}(\mu_i) + l_i x_2,\ i = 2, \ldots, r\}$ and $0 = m_1 x_1 \geq \max\{\mathrm{val}(\nu_i) + n_i x_2,\ i = 2, \ldots, s\}$, and thus if v is the closest endpoint of ξ to the origin of $\mathbb{R}^2$, then the second coordinate κ of v is equal to

$$\max\{x_2 \mid 0 = \max\{\mathrm{val}(\mu_i) + l_i x_2\ ,\ \mathrm{val}(\nu_i) + n_i x_2 \mid i = 2, \ldots, r\,,\, i = 2, \ldots, s\}\}.$$

Similarly, we show that the second coordinate κ_0 of v_0 ($\kappa_0 = -\infty$ if $\mathfrak{E}$ is unbounded) is equal to

$$\min\{x_2 \mid 0 = \max\{\mathrm{val}(\mu_i) + l_i x_2\ ,\ \mathrm{val}(\nu_i) + n_i x_2 \mid i = 2, \ldots, r\,,\, i = 2, \ldots, s\}\}.$$

□

Remark 6.7. *We have the following:*

- ***a)*** *Since the transformations from* (6.2.1) *to* (6.2.2) *are a series of change of coordinates, condition **ii)** of Proposition 6.6 gives a bijection between the set of non-degenerate positive solutions of* (6.2.1) *with valuation in* $\overset{\circ}{\xi}$*, and the set of such solutions with valuations in* $\overset{\circ}{\mathfrak{E}}$*.*

- ***b)*** *If* $(\alpha, \beta) \in (\mathbb{K}^*)^2$ *is a non-degenerate solution of* (6.2.2) *with* $\mathrm{Val}(\alpha, \beta)$ *in* $\overset{\circ}{\mathfrak{E}}$*, then condition **i)** of Proposition 6.6 implies that* $\mathrm{coef}(\alpha) = 1$ *and* $\mathrm{ord}(\alpha) = 0$*. Thus, to determine* $\mathrm{Val}(\alpha, \beta)$ *and* $\mathrm{Coef}(\alpha, \beta)$*, it remains to determine* $\mathrm{val}(\beta)$ *and* $\mathrm{coef}(\beta)$*. This is the purpose of Proposition 6.8.*

Thanks to Proposition 6.6, we are interested in non-degenerate positive solutions of (6.2.2) with valuation in $\overset{\circ}{\mathfrak{E}} \subset \{0\} \times\,]-\infty, 0[$. We also assume that (6.2.2) satisfies property **i)** of Proposition 6.6. Consider the polynomial

$$A(y)/k_1 - B(y)/m_1 \tag{6.2.4}$$

with

$$A(y) = \mathrm{coef}(c_0 + c_1) t^{\mathrm{ord}(c_0 + c_1)} + \sum_{i=2}^{r} \mathrm{coef}(c_i) t^{\mathrm{ord}(c_i)} y^{l_i}$$

and

$$B(y) = \mathrm{coef}(d_0 + d_1) t^{\mathrm{ord}(d_0 + d_1)} + \sum_{i=2}^{s} \mathrm{coef}(d_i) t^{\mathrm{ord}(d_i)} y^{n_i}.$$

Proposition 6.8. *If* $(\alpha, \beta) \in (\mathbb{RK}^*)^2$ *is a non-degenerate solution of* (6.2.2) *such that* $\mathrm{ord}(\alpha) = 0$ *and* $\mathrm{coef}(\alpha) = 1$*, then there exists a non-degenerate root* $\gamma \in \mathbb{RK}^*$ *of* (6.2.4) *such that* $\mathrm{ord}(\gamma) = \mathrm{ord}(\beta)$ *and* $\mathrm{coef}(\gamma) = \mathrm{coef}(\beta)$*.*

Proof. Assume that $(\alpha, \beta) \in (\mathbb{RK}^*)^2$ is a non-degenerate solution of (6.2.2) such that $\mathrm{ord}(\alpha) = 0$ and $\mathrm{coef}(\alpha) = 1$. Then $\alpha = 1 + \delta$ with $\delta \in \mathbb{RK}$ and $\mathrm{ord}(\delta) > 0$. Replacing y_1 by $1 + x$ and y_2 by y, the system (6.2.2) becomes

$$P(x, y) = Q(x, y) = 0, \tag{6.2.5}$$

where

$$P(x,y) = c_0 + c_1 + \sum_{i=1}^{k_1} c_1 \binom{k_1}{i} x^i + \sum_{i=2}^{r} c_i (1+x)^{k_i} y^{l_i}$$

and

$$Q(x,y) = d_0 + d_1 + \sum_{j=1}^{m_1} d_1 \binom{m_1}{j} x^i + \sum_{i=2}^{s} d_i (1+x)^{m_i} y^{n_i}.$$

Set $a_i = c_i$ for $i = 2, \ldots, r$ and $a_1 = c_0 + c_1$. Similarly, set $b_i = d_i$ for $i = 2, \ldots, s$ and $b_1 = d_0 + d_1$. Then (6.2.5) becomes

$$\begin{array}{lllll} \sum_{i=1}^{k_1} c_1 \binom{k_1}{i} x^i & + & a_1 & + & \sum_{i=2}^{r} a_i (1+x)^{k_i} y^{l_i} = 0, \\ \sum_{i=1}^{m_1} d_1 \binom{m_1}{i} x^i & + & b_1 & + & \sum_{i=2}^{s} b_i (1+x)^{m_i} y^{n_i} = 0. \end{array} \tag{6.2.6}$$

From $\mathrm{ord}(\delta) > 0$, we deduce that

$$a_1 + \sum_{i=2}^{r} a_i (1+\delta)^{k_i} \beta^{l_i} \quad \text{and} \quad b_1 + \sum_{i=2}^{s} b_i (1+\delta)^{m_i} \beta^{n_i}$$

have the same order as

$$A(\beta) = \sum_{i=1}^{r} \mathrm{coef}(a_i) t^{\mathrm{ord}(a_i)} \beta^{l_i} \quad \text{and} \quad B(\beta) = \sum_{i=1}^{s} \mathrm{coef}(b_i) t^{\mathrm{ord}(b_i)} \beta^{n_i}$$

respectively, where $l_1 = n_1 = 0$.

Consider the two polynomials g, h in $\mathbb{RK}[x]$ defined by $g(x) = k_1(c_1 - 1)x + \sum_{i=2}^{k_1} c_1 \binom{k_1}{i} x^i$ and $h(x) = m_1(d_1 - 1)x + \sum_{i=2}^{m_1} d_1 \binom{m_1}{i} x^i$ so that

$$\sum_{i=1}^{k_1} c_1 \binom{k_1}{i} x^i = k_1 x + g(x) \quad \text{and} \quad \sum_{j=1}^{m_1} d_1 \binom{m_1}{i} x^i = m_1 x + h(x).$$

Set $\mathrm{ord}(\beta) = \beta_0$. Then $M = \min\{l_i\beta_0 + \mathrm{ord}(a_i),\ i = 1, \ldots, r\}$ is the order of $A(\beta)$. Similarly, $N = \min\{\beta_0 n_i + \mathrm{ord}(b_i),\ i = 1, \ldots, s\}$ is the order of $B(\beta)$. Denote by I (resp. J) the set $\{i \in [r] \mid l_i\beta_0 + \mathrm{ord}(a_i) = M\}$ (resp. $\{i \in [s] \mid n_i\beta_0 + \mathrm{ord}(b_i) = N\}$).

Plugging $(t^{\mathrm{ord}(\delta)}x, t^{\beta_0}y)$ in (6.2.6), and dividing its first and second equation by $k_1 t^M$ and $m_1 t^N$ respectively will not change the number of its solutions in $\mathbb{RK} \times \mathbb{RK}^*$. Expanding both polynomials of (6.2.6) in terms of x and y gives

$$\begin{array}{lllllll} t^{\mathrm{ord}(\delta)-M}x & + & t^{-M} g(t^{\mathrm{ord}(\delta)}x)/k_1 & + & \sum_{i\in I} \mathrm{coef}(a_i/k_1) y^{l_i} & + & G(x,y) = 0, \\ t^{\mathrm{ord}(\delta)-N}x & + & t^{-N} h(t^{\mathrm{ord}(\delta)}x)/m_1 & + & \sum_{i\in J} \mathrm{coef}(b_i/m_1) y^{n_i} & + & H(x,y) = 0, \end{array} \tag{6.2.7}$$

where all the coefficients of the polynomials G and H of $\mathbb{RK}[x^{\pm 1}, y^{\pm 1}]$ have positive orders. Note that the polynomials g and h have coefficients with non-negative orders. Indeed, since $\mathrm{ord}(c_1) = \mathrm{ord}(d_1) = 0$ and $\mathrm{coef}(c_1) = \mathrm{coef}(d_1) = 1$, we have $\mathrm{ord}(c_1 - 1) > 0$ and $\mathrm{ord}(d_1 - 1) > 0$.

Doing slight perturbations on the coefficients of (6.2.4), we may assume without loss of generality that the polynomial (6.2.4) has only non-degenerate roots in $\mathbb{RK}^*$, and that for any $I \subset [r]$, $J \subset [s]$ the polynomials $\sum_{i\in I} \mathrm{coef}(a_i)y^{l_i}$ and $\sum_{i\in J} \mathrm{coef}(b_i)y^{n_i}$ don't have a non-zero common root. Such perturbations do not change the number of non-degenerate solutions of (6.2.5) in $\mathbb{RK} \times \mathbb{RK}^*$ nor do they change the number of non-degenerate roots of (6.2.4) in $\mathbb{RK}^*$. We have that at least one of $\mathrm{ord}(\delta) - M$ and $\mathrm{ord}(\delta) - N$ is equal to zero and none of them can be negative.

Indeed, assume first that both of them are positive. Note that from $\mathrm{ord}(\delta) > 0$, we have $\min(\mathrm{ord}(g(\delta)), \mathrm{ord}(h(\delta))) > \mathrm{ord}(\delta)$ if $\delta \neq 0$. Moreover, since $(\delta, \beta) \in \mathbb{RK}\times\mathbb{RK}^*$ is a non-degenerate solution of (6.2.6), for $t > 0$ small enough, we have that $\mathrm{coef}(\beta)$ is a real non-degenerate solution of

$$\sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i} = \sum_{i\in J} \mathrm{coef}(b_i/m_1)y^{n_i} = 0,$$

a contradiction. Assume now that we have for example $\mathrm{ord}(\delta)-M$ is negative. Divide the first equation of (6.2.7) by $t^{\mathrm{ord}(\delta)-M}$. Then we get terms $t^{-\mathrm{ord}(\delta)}g(t^{\mathrm{ord}(\delta)}x)/k_1$, $t^{M-\mathrm{ord}(\delta)}\sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i}$ and $t^{M-\mathrm{ord}(\delta)}G(x, y)$ which tend to zero when $t \to 0$. This proves that $\mathrm{coef}(\delta) = 0$, which means that $\delta = 0$. It follows that $\mathrm{coef}(\beta)$ is a non-degenerate real solution of

$$\sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i} = \sum_{i\in J} \mathrm{coef}(b_i/m_1)y^{n_i} = 0,$$

a contradiction.

We conclude that δ is non-zero and we study two cases.

i) First case: $M = N = \mathrm{ord}(\delta)$. Since $(\delta, \beta) \in (\mathbb{RK}^*)^2$ is a solution of (6.2.6), taking $t > 0$ small enough, we get that $(\mathrm{coef}(\delta), \mathrm{coef}(\beta))$ is a real solution of

$$x + \sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i} = x + \sum_{i\in J} \mathrm{coef}(b_i/m_1)y^{n_i} = 0. \tag{6.2.8}$$

Taking the difference of the two non-zero polynomials appearing in (6.2.8), we deduce that $\mathrm{coef}(\beta)$ is a real root of

$$\sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i} - \sum_{i\in J} \mathrm{coef}(b_i/m_1)y^{n_i}.$$

On the other hand, we have

$$A(t^{\beta_0}y)/(k_1t^{\mathrm{ord}(\delta)}) = \sum_{i\in I} \mathrm{coef}(a_i/k_1)y^{l_i} + \sum_{i\notin I} \mathrm{coef}(a_i/k_1)t^{\beta_0 l_i+\mathrm{ord}(a_i)-\mathrm{ord}(\delta)}y^{l_i}$$

and

$$B(t^{\beta_0}y)/(m_1t^{\mathrm{ord}(\delta)}) = \sum_{i\in J} \mathrm{coef}(b_i/m_1)y^{n_i} + \sum_{i\notin J} \mathrm{coef}(b_i/m_1)t^{\beta_0 n_i+\mathrm{ord}(b_i)-\mathrm{ord}(\delta)}y^{n_i}.$$

Consequently, $A(t^{\beta_0}y)/(k_1t^{\mathrm{ord}(\delta)}) - B(t^{\beta_0}y)/(m_1t^{\mathrm{ord}(\delta)})$ has a root $\rho \in \mathbb{RK}^*$ with $\mathrm{ord}(\rho) = 0$ and $\rho(0) = \mathrm{coef}(\beta)$, and thus, $\gamma = t^{\beta_0}\rho$ is a root of (6.2.4).

ii) Second case: $\text{ord}(\delta) = N > M$ (the case where $\text{ord}(\delta) = M > N$ is symmetric). Similarly, since $(\delta, \beta) \in (\mathbb{RK}^*)^2$ is a solution of (6.2.6), when $t > 0$ is small enough, we have that $(\text{coef}(\delta), \text{coef}(\beta))$ is a real solution of

$$\sum_{i \in I} \text{coef}(a_i/k_1) y^{l_i} = x + \sum_{i \in J} \text{coef}(b_i/m_1) y^{n_i} = 0. \tag{6.2.9}$$

On the other hand, all coefficients of $t^{-M} B(t^{\beta_0} y)$ have positive order. Indeed, since $M < N$, we have $\text{ord}(b_i) + n_i \beta_0 - M > 0$ for $i = 1, \ldots, s$. Consequently,

$$\sum_{i \in I} \text{coef}(a_i/k_1) y^{l_i} + \sum_{i \notin I} \text{coef}(a_i/k_1) t^{\beta_0 l_i + \text{ord}(a_i) - M} y^{l_i} - t^{-M} B(t^{\beta_0} y)/m_1$$

has a root $\rho \in \mathbb{RK}^*$ with $\text{ord}(\rho) = 0$ and $\rho(0) = \text{coef}(\beta)$. Therefore, $\gamma = t^{\beta_0} \rho$ is a root of (6.2.4).

□

Similarly to the one that appeared in Chapter 5, the polynomial f_t defined by the equation in (6.2.4) is a particular case of a *Viro polynomial* (c.f. [BBS06, Bih02, Vir84]). We recall now the description for f_t that was made in Section 5.3 of Chapter 5.

Write $f_t(y) = \sum_{p=p_0}^{d} \phi_p(t) y^p$, where t is a positive real number, and each coefficient $\phi_p(t)$ is a finite sum $\sum_{q \in I_p} c_{p,q} t^q$ with $c_{p,q} \in \mathbb{R}$ and q a real number. Write f for the function of y and t defined by f_t. Let $D \subset \mathbb{R}^2$ be the convex hull of the points (p, q) for $p_0 \leq p \leq d$ and $q \in I_p$. Assume that D has dimension 2. Its lower hull Γ is the union of the edges $e_1, \ldots, e_l$ of D whose inner normals have positive second coordinate. Let I_i be the image of e_i under the projection $\mathbb{R}^2 \to \mathbb{R}$ forgetting the last coordinate. Then the intervals $I_1, \ldots, I_l$ subdivide the Newton segment $[p_0, d]$ of f_t. Let $f^{(i)}$ be the facial subpolynomial of f for the face e_i. That is, $f^{(i)}$ is the sum of terms $c_{p,q} y^p$ such that $(p, q) \in e_i$. Suppose that e_i is the graph of $y \mapsto \lambda_i y + \mu_i$ over I_i. Expanding $f_t(y t^{-\lambda_i})/t^{\mu_i}$ in powers of t gives

$$f_t(y t^{-\lambda_i})/t^{\mu_i} = f^{(i)}(y) + g_t^{(i)}(y) \quad \text{and} \quad i = 1, \ldots, l, \tag{6.2.10}$$

where $g_t^{(i)} \in \mathbb{RK}[y]$ collects the terms whose powers of t are positive. Then $f^{(i)}(y)$ has Newton segment I_i and its number of non-degenerate non-zero roots in $\mathbb{K}$ counted with multiplicities is $|I_i|$, the integer length of the interval I_i.

Definition 6.9. *An element y_0 in $\mathbb{K}^*$ is **largely ordered** with respect to $f_t = \sum_{p=p_0}^{d} \phi_p(t) y^p$ if $p \cdot \text{ord}(y_0) + \text{ord}(\phi_p(t)) > 0$ for $p = p_0, \ldots, d$.*

Recall that we are interested in the number of non-degenerate positive solutions $(\alpha, \beta) \in (\mathbb{RK}^*)^2$ of (6.2.2) such that $\text{Val}(\alpha, \beta) \in \mathring{\mathfrak{C}} =](0, \kappa_0), (0, \kappa)[$. By Proposition 6.8, this number is bounded by the number of non-degenerate positive roots γ of the polynomial f_t appearing in (6.2.4) which satisfy $\text{val}(\gamma) \in]\kappa_0, \kappa[$.

Lemma 6.10. *If $\text{Val}(\alpha, \beta) \in \mathring{\mathfrak{C}}$ for some $(\alpha, \beta) \in (\mathbb{RK}^*)^2$, then β is largely ordered with respect to f_t.*

Proof. Recall that f_t is defined by (6.2.4). Assume that $\mathrm{Val}(\alpha,\beta) \in \mathring{\mathfrak{E}}$ for some $(\alpha,\beta) \in (\mathbb{RK}^*)^2$. Then since $\mathfrak{E} \subset \{0\} \times \mathbb{R}$, we have $\mathrm{val}(\alpha) = 0$. Moreover, $\mathrm{val}(\beta)$ satisfies $0 > \max_{i=2}^r\{\mathrm{val}(c_i) + l_i \mathrm{val}(\beta)\}$ and $0 > \max_{i=2}^r\{\mathrm{val}(d_j) + n_j \mathrm{val}(\beta)\}$. Indeed, from condition **i)** of Proposition 6.6, we have that $\mathrm{Val}(\alpha,\beta)$ belongs to the relative interior of the duals of $[0,k_1]$ and $[0,m_1]$. Therefore, β is largely ordered from $\mathrm{val}(c_i) + l_i \mathrm{val}(\beta) = -\mathrm{ord}(c_i) - l_i \mathrm{ord}(\beta)$ and $\mathrm{val}(d_j) + n_j \mathrm{val}(\beta) = -\mathrm{ord}(d_j) - n_j \mathrm{ord}(\beta)$. □

Doing perturbations on the coefficients appearing in the polynomials $f^{(i)}$, we may assume that for $i = 1,\dots,l$, the roots of $f^{(i)}$ are non-degenerate. Recall equation (6.2.10) relating f_t to the facial subpolynomials f_i.

Lemma 6.11. *If γ is largely ordered with respect to f_t and a non-degenerate non-zero root of f_t, then there exists $i \in [l]$ such that $\mathrm{coef}(\gamma)$ is a non-degenerate non-zero root of $f^{(i)}$, $\mathrm{val}(\gamma) = \lambda_i$ and $\mu_i > 0$. This induces a bijection between the set of largely ordered non-degenerate non-zero roots γ of f_t and the set of non-degenerate non-zero roots of the polynomials $f^{(i)}$ such that $\mu_i > 0$.*

Proof. Assume that γ is a largely ordered non-degenerate root of f_t with $\mathrm{ord}(\gamma) = \beta_0$ and $\mathrm{coef}(\gamma) = \rho_0 \neq 0$. Write $f_t(t^{\beta_0}y)$ as

$$f_t(t^{\beta_0}y) = t^{\delta}(r(y) + s_t(y)) \tag{6.2.11}$$

for some $\delta \in \mathbb{R}$, $r \in \mathbb{R}[y]$ and $s_t \in \mathbb{RK}[y]$, where all exponents of t in $s_t(y)$ are positive. Then the Newton polytope of $t^{\delta}r(y)$ is a face of the Newton polytope of $f_t(t^{\beta_0}y)$. Since γ is a non-zero root of f_t with $\mathrm{ord}(\gamma) = \beta_0$, the polynomial $f_t(t^{\beta_0}y)$ has a non-zero root y_0 with $\mathrm{ord}(y_0) = 0$. It follows that $\rho_0 = \mathrm{coef}(y_0)$ is a non-zero root of $r(y)$, and thus $r(y)$ has at least two terms (its Newton polytope is a segment). The Newton polytope of $f_t(t^{\beta_0}y)$ is obtained from that of $f_t(y)$ by a linear map $(a,b) \mapsto (a, b+\beta_0 a)$. Note that such linear map (independent of $\beta_0 \in \mathbb{R}$) maps a lower face to a lower face. Comparing with (6.2.10), we obtain that there exists $i \in [l]$ such that $r = f^{(i)}$, $s = g^{(i)}$, $\beta_0 = -\lambda_i$ and $\delta = \mu_i$. Therefore, when $t > 0$ is small enough, $t^{\lambda_i}\gamma$ is close to a non-degenerate root of $f^{(i)}(y)$. Let M be the minimum of the quantities $p\,\mathrm{ord}(t^{\beta_0}y) + \mathrm{ord}(\phi_p(t))$, $p = 0,\dots,d$. Then $M > 0$ since γ is largely ordered. Now $f_t(t^{\beta_0}y) = \sum_{p=p_0}^{d} \phi_p(t) t^{p\beta_0} y^p$ with $\mathrm{ord}(\phi_p(t)t^{p\beta_0}) \geq M$ and there is at least one equality. Comparing with (6.2.11), we get $M = \delta$ and thus $\mu_i = M > 0$.

Assume that ρ_0 is a non-degenerate non-zero root of $f^{(i)}$ and μ_i is positive. Then (6.2.10) will have a root $\rho \in \mathbb{RK}^*$ with $\mathrm{ord}(\rho) = 0$ and $\rho(0) = \rho_0$ for $t > 0$ small enough. Therefore, $\gamma = t^{-\lambda_i}\rho$ is a non-degenerate root of f_t. Finally, γ is largely ordered since $\mu_i > 0$. □

If (α,β) is a solution of (6.2.2) such that $\mathrm{Val}(\alpha,\beta) \in \mathring{\mathfrak{E}}$, then $\alpha = 1 + x$ with $\mathrm{ord}(x) > 0$. Plugging $(1+x,\beta)$ in (6.2.2), gives a polynomial system in (x,β) which does not depend on $\mathrm{coef}(c_0)$, $\mathrm{coef}(c_1)$, $\mathrm{coef}(d_0)$ or $\mathrm{coef}(d_1)$. This follows from $\mathrm{coef}(c_0) = \mathrm{coef}(d_0) = -1$, $\mathrm{coef}(c_1) = \mathrm{coef}(d_1) = 1$, $\mathrm{ord}(c_0) = \mathrm{ord}(d_0) = \mathrm{ord}(c_1) = \mathrm{ord}(d_1) = 0$ (see Proposition 6.6). Therefore, perturbing slightly $c_2,\dots,c_r$, $d_2,\dots,d_s$ and the non-constant terms of c_0, d_0, c_1, d_1, we may assume without loss of generality that if (α,β) and (α',β') are two different solutions of (6.2.2) with valuations in $\mathfrak{E}$, then $\mathrm{coef}(\beta) \neq \mathrm{coef}(\beta')$. Obviously, such a perturbation does not change the number of non-degenerate positive solutions of (6.2.2). It follows from Proposition 6.8 that the set of positive solutions (α,β) of (6.2.2) with $\mathrm{Val}(\alpha,\beta) \in \mathring{\mathfrak{E}}$ is mapped injectively to the set of positive roots γ of (6.2.4). Set

$$\mathcal{I} := \{y \in \mathbb{R}_{>0} \mid \exists i \in [l]\ ;\ f_i(y) = 0,\ \lambda_i \in]\kappa_0,\kappa[,\ \mu_i > 0\}.$$

We have the following Corollary.

Corollary 6.12. *If* $(\alpha, \beta) \in (\mathbb{RK}_{>0})^2$ *is a non-degenerate solution of* (6.2.2) *such that* $\mathrm{Val}(\alpha, \beta) \in \mathring{\mathfrak{E}}$, *then* $\mathrm{coef}(\alpha) = 1$, $\mathrm{ord}(\alpha) = 0$ *and for some* $i \in [l]$, *we have* $f_i(\mathrm{coef}(\beta)) = 0$ *and* $\mathrm{ord}(\beta) = -\lambda_i$. *This induces an injection* $(z_1, z_2) \mapsto \mathrm{coef}(z_2)$ *from the set of non-degenerate positive solutions of* (6.2.2) *with valuation in* $\mathfrak{E}$, *onto the set* $\mathcal{I}$.

Proof. It is clear from before that if (6.2.2) has a solution $(\alpha, \beta) \in (\mathbb{RK}^*)^2$ with valuation in $\mathring{\mathfrak{E}}$, then $\mathrm{coef}(\alpha) = 1$ and $\mathrm{ord}(\alpha) = 0$.

Proposition 6.8 and Lemma 6.10 show that the set of non-degenerate solutions $(\alpha, \beta) \in (\mathbb{RK}_{>0})^2$ of (6.2.2) such that $\mathrm{Val}(\alpha, \beta) \in \mathring{\mathfrak{E}}$ is mapped injectively onto the set of non-degenerate roots γ of f_t that are largely ordered with respect to f_t and that satisfy $\mathrm{ord}(\gamma) = \mathrm{ord}(\beta) \in]-\kappa, -\kappa_0[$ and $\mathrm{coef}(\gamma) = \mathrm{coef}(\beta)$. Moreover, Lemma 6.11 shows that the set of such roots γ is in bijection with the set of positive non-degenerate non-zero roots ρ_0 of the polynomials $f^{(i)}$ such that $\lambda_i = -\mathrm{ord}(\gamma)$, $\mathrm{coef}(\gamma) = \rho_0$ and $\mu_i > 0$. $\square$

Definition 6.13. *We say that the polynomial* f_t *in* (6.2.4) *is an* ***approximation polynomial*** *of* (6.2.1) *for* ξ.

Now, to sum up this Section. One can approximate combinatorically all non-degenerate positive solutions of (6.2.1) with valuation contained in the relative interior $\mathring{\xi}$ of a cell ξ of type (I) by computing their first-order terms. In order to achieve that, Proposition 6.6 shows that it suffices to determine the first-order terms of the non-degenerate solutions $(\alpha, \beta) \in (\mathbb{RK}_{>0})^2$ of (6.2.2) with valuation in $\mathring{\mathfrak{E}}$ (see Figure 6.2). The first-order term of α is $1 \cdot t^0$, and by Corollary 6.12, there exists $i \in [l]$ such that $f_i(\beta) = 0$, $\mathrm{ord}(\beta) = -\lambda_i$ and $\mu_i > 0$. The numbers λ_i and μ_i are determined from the lower hull of D (the Newton polytope of $g(y, t) := f_t(y)$).

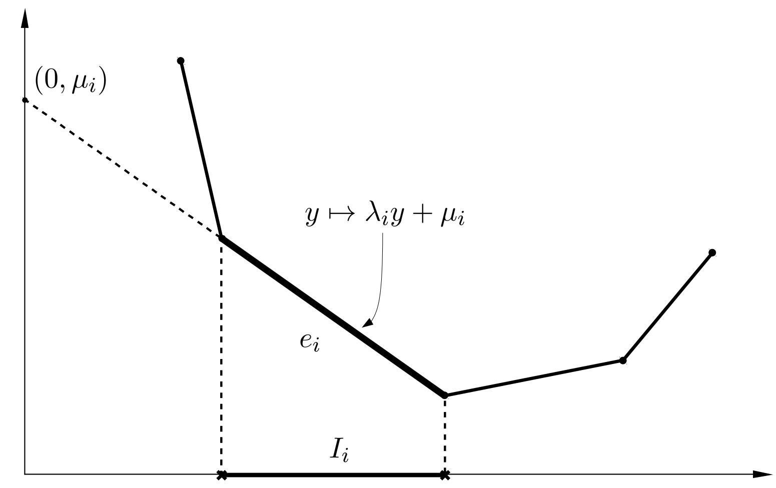

Figure 6.2: Lower part of D associated to f_t: here, $\lambda_i < 0$ and $\mu_i > 0$

6.3 Base fans and tropical intersections

In this section, we consider a system defined on the field of real generalized locally convergent Puiseux series with two equations in two variables supported on a set of five distinct points in $\mathbb{Z}^2$.

We say that such system is of type $n = k = 2$. Moreover, we assume that no three points of the support belong to a line. We say that such a system is highly non-degenerate.

Lemma 6.14. *Given any system of polynomials in* $\mathbb{RK}[z_1^{\pm 1}, z_2^{\pm 1}]$ *of type* $n = k = 2$*, one can associate to it a system*

$$\begin{aligned} a_0 z^{w_0} + a_1 z^{w_1} + a_2 z^{w_2} + a_3 t^{\alpha} z^{w_3} &= 0, \\ b_0 z^{w_0} + b_1 z^{w_1} + b_2 z^{w_2} + b_4 t^{\beta} z^{w_4} &= 0, \end{aligned} \tag{6.3.1}$$

with equations in $\mathbb{RK}[z_1^{\pm 1}, z_2^{\pm 1}]$*, that has the same number of positive non-degenerate solutions, where all* a_i *and* b_j *are in* $\in \mathbb{RK}^*$ *and verify* $\mathrm{ord}(a_i) = \mathrm{ord}(b_j) = 0$*, all* w_i *are in* $\mathbb{Z}^2$ *and both* α*,* β *are real numbers.*

Proof. Using linear combinations, any system of type $n = k = 2$ can be reduced to a system

$$\begin{aligned} c_0 t^{\alpha_0} z^{\tilde{w}_0} + c_1 t^{\alpha_1} z^{\tilde{w}_1} + c_2 t^{\alpha_2} z^{\tilde{w}_2} + c_3 t^{\alpha_3} z^{\tilde{w}_3} &= 0, \\ d_0 t^{\beta_0} z^{\tilde{w}_0} + d_1 t^{\beta_1} z^{\tilde{w}_1} + d_2 t^{\beta_2} z^{\tilde{w}_2} + d_4 t^{\beta_4} z^{\tilde{w}_4} &= 0 \end{aligned} \tag{6.3.2}$$

that has the same number of positive non-degenerate solutions, where all c_i and d_j are in $\in \mathbb{RK}^*$ and verify $\mathrm{ord}(c_i) = \mathrm{ord}(d_j) = 0$, all $\tilde{w}_i$ are in $\mathbb{Z}^2$ and all exponents of t are real numbers. Assume first that $\alpha_i - \alpha_1 \neq \beta_i - \beta_1$ for $i = 0, 2$. By symmetry, the different possibilities of inequalities can be reduced to only two cases.

- First case: $\alpha_0 - \alpha_1 < \beta_0 - \beta_1$ and $\alpha_2 - \alpha_1 < \beta_2 - \beta_1$.
  Since we are interested in non-degenerate positive solutions, we may suppose that $\tilde{w}_0 = (0, 0)$. The system

  $$\begin{aligned} (c_0/c_1) t^{\alpha_0 - \alpha_1} z^{\tilde{w}_0} + z^{\tilde{w}_1} + (c_2/c_1) t^{\alpha_2 - \alpha_1} z^{\tilde{w}_2} + (c_3/c_1) t^{\alpha_3 - \alpha_1} z^{\tilde{w}_3} &= 0, \\ \tilde{c}_0 t^{\alpha_0 - \alpha_1} z^{\tilde{w}_0} + \tilde{c}_2 t^{\alpha_2 - \alpha_1} z^{\tilde{w}_2} + (c_3/c_1) t^{\alpha_3 - \alpha_1} z^{\tilde{w}_3} - (d_4/d_1) t^{\beta_4 - \beta_1} z^{\tilde{w}_4} &= 0 \end{aligned} \tag{6.3.3}$$

  has the same number of non-degenerate positive solutions as (6.3.2). Indeed, the first equation of (6.3.3) is obtained by dividing the first equation of (6.3.2) by $c_1 t^{\alpha_1}$, whereas the second equation of (6.3.3) is obtained by dividing the first equation of (6.3.2) by $c_1 t^{\alpha_1}$ and subtracting from it the second equation of (6.3.2) divided by $d_1 t^{\beta_1}$. Note that $\mathrm{coef}(\tilde{c}_i) = \mathrm{coef}(c_i/c_1)$ and $\mathrm{ord}(\tilde{c}_1) = 0$ for $i = 0, 2$. We divide both equations of (6.3.3) by $t^{\alpha_0 - \alpha_1}$ and set $w_3 = \tilde{w}_1$, $w_2 = \tilde{w}_3$, $w_1 = \tilde{w}_2$ and $w_i = \tilde{w}_i$ for $i = 0, 4$. Finally replacing (z_1, z_2) by $(t^k z_1, t^l z_2)$ in (6.3.3) for some real numbers k and l satisfying $\langle (k, l), w_2 \rangle = \alpha_0 - \alpha_3$ and $\langle (k, l), w_1 \rangle = \alpha_0 - \alpha_2$ does not change the number of positive non-degenerate solutions of (6.3.3). This gives a system of the form (6.3.1) with the same number of non-degenerate positive solutions as (6.3.2).

- Second case: $\alpha_0 - \alpha_1 < \beta_0 - \beta_1$ and $\alpha_2 - \alpha_1 > \beta_2 - \beta_1$.
  Note that this case gives $\alpha_2 - \alpha_0 > \beta_2 - \beta_0$. Since we are interested in non-degenerate positive solutions, we may suppose that $\tilde{w}_4 = (0, 0)$. The system

  $$\begin{aligned} (d_1/d_0) t^{\beta_1 - \beta_0} z^{\tilde{w}_1} + (d_2/d_0) t^{\beta_2 - \beta_0} z^{\tilde{w}_2} + (d_4/d_0) t^{\beta_4 - \beta_0} z^{\tilde{w}_4} + z^{\tilde{w}_0} &= 0, \\ \tilde{d}_1 t^{\beta_1 - \beta_0} z^{\tilde{w}_1} + \tilde{d}_2 t^{\beta_2 - \beta_0} z^{\tilde{w}_2} - (c_3/c_0) t^{\alpha_3 - \alpha_0} z^{\tilde{w}_3} + (d_4/d_0) t^{\beta_4 - \beta_0} z^{\tilde{w}_4} &= 0 \end{aligned} \tag{6.3.4}$$

has the same number of non-degenerate positive solutions as (6.3.2). Indeed, the first equation of (6.3.4) is obtained by dividing the second equation of (6.3.2) by $d_0t^{\beta_0}$, whereas the second equation of (6.3.4) is obtained by dividing the second equation of (6.3.2) by $d_0t^{\beta_0}$ and subtracting from it the first equation of (6.3.2) divided by $c_0t^{\alpha_0}$. Note that $\mathrm{coef}(\tilde{d}_i) = \mathrm{coef}(d_i/d_0)$ and $\mathrm{ord}(\tilde{d}_i) = 0$ for $i = 1, 2$. We divide both equations of (6.3.4) by $t^{\beta_4-\beta_0}$ and set $w_0 = \tilde{w}_4$, $w_4 = \tilde{w}_0$ and $w_i = \tilde{w}_i$ for $i = 1, 2, 3$. Finally replacing (z_1, z_2) by $(t^k z_1, t^l z_2)$ in (6.3.4) for some real numbers k and l satisfying $\langle (k,l), w_1\rangle = \beta_4 - \beta_1$ and $\langle (k,l), w_2\rangle = \beta_4 - \beta_2$ does not change the number of positive non-degenerate solutions of (6.3.5). This gives a system of the form (6.3.1) with the same number of non-degenerate positive solutions as (6.3.2).

Assume now that we have $\alpha_i - \alpha_1 = \beta_i - \beta_1$ for either $i = 0$ or $i = 2$. The case where we have equality for both $i = 0$ and $i = 2$ is trivial. Without loss of generality, we may suppose that $\alpha_0 - \alpha_1 = \beta_0 - \beta_1$ and $\alpha_2 - \alpha_1 < \beta_2 - \beta_1$. Note that this case gives $\beta_0 - \beta_2 < \alpha_0 - \alpha_2$. Since we are interested in non-degenerate positive solutions, we may suppose that $\tilde{w}_0 = (0, 0)$. The system

$$\begin{aligned} (d_0/d_2)t^{\beta_0-\beta_2}z^{\tilde{w}_0} + (d_1/d_2)t^{\beta_1-\beta_2}z^{\tilde{w}_1} + z^{\tilde{w}_2} + (d_4/d_2)t^{\beta_4-\beta_2}z^{\tilde{w}_4} &= 0, \\ \tilde{d}_0t^{\beta_0-\beta_2}z^{\tilde{w}_0} + \tilde{d}_1t^{\beta_1-\beta_2}z^{\tilde{w}_1} - (c_3/c_2)t^{\alpha_3-\alpha_0}z^{\tilde{w}_3} + (d_4/d_2)t^{\beta_4-\beta_2}z^{\tilde{w}_4} &= 0 \end{aligned} \qquad (6.3.5)$$

has the same number of non-degenerate positive solutions of (6.3.2). Indeed, the first equation of (6.3.5) is obtained by dividing the second equation of (6.3.2) by $d_2t^{\beta_2}$, whereas the second equation of (6.3.5) is obtained by dividing the second equation of (6.3.2) by $d_2t^{\beta_2}$ and subtracting from it the first equation of (6.3.2) divided by $c_2t^{\alpha_2}$. Note that $\mathrm{coef}(\tilde{d}_i) = \mathrm{coef}(d_i/d_2)$ and $\mathrm{ord}(\tilde{d}_i) = 0$ for $i = 0, 1$. We divide both equations of (6.3.5) by $t^{\beta_0-\beta_2}$ and set $w_2 = \tilde{w}_4$, $w_4 = \tilde{w}_2$ and $w_i = \tilde{w}_i$ for $i = 0, 1, 3$. Finally replacing (z_1, z_2) by $(t^k z_1, t^l z_2)$ in (6.3.5) for some real numbers k and l satisfying $\langle (k,l), w_1\rangle = \beta_1 - \beta_0$ and $\langle (k,l), w_2\rangle = \beta_4 - \beta_0$ does not change the number of positive non-degenerate solutions of (6.3.5). This gives a system of the form (6.3.1) with the same number of non-degenerate positive solutions as (6.3.2). □

Consider a system (6.3.1) satisfying all the hypotheses of Lemma 6.14. Since we are interested in its non-degenerate positive solutions, we may assume that $w_0 = (0, 0)$. Moreover, without loss of generality, we may assume that $a_1 = b_1 = 1$. For the simplicity of further computations, we make the following change of coordinates. Let m_1 be the greatest common divisor of the coordinates of w_1. Setting $y_1 = z^{\frac{w_1}{m_1}}$ and choosing any basis of $\mathbb{Z}^2$ with first vector $\frac{1}{m_1}\cdot w_3$, we get a monomial change of coordinates $(z_1, z_2) \mapsto (y_1, y_2)$ of $(\mathbb{RK}^*)^2$ such that $z^{w_1} = y_1^{m_1}$ and $z^{w_2} = y_1^{m_2}y_2^{n_2}$. Replacing y_2 by y_2^{-1} if necessary, we assume that $n_2 > 0$. Indeed, $n_2 \neq 0$, since by assumption the support of (6.3.1) is highly non-degenerate. With respect to these new coordinates, the system (6.3.1) becomes the following normalized system (see Section 6.1 for the definition).

$$\begin{aligned} a_0 + y_1^{m_1} + a_2y_1^{m_2}y_2^{n_2} + a_3t^{\alpha}y_1^{m_3}y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2y_1^{m_2}y_2^{n_2} + b_4t^{\beta}y_1^{m_4}y_2^{n_4} &= 0. \end{aligned} \qquad (6.3.6)$$

Note that (6.3.1) and (6.3.6) have the same number of positive solutions. Later, we denote by w_i the vector (m_i, n_i) in (6.3.6).

Let T_1 (resp. Δ_1, τ_1) denote the tropical hypersurface (resp. the Newton polytope, the dual subdivision of the Newton polytope) associated to the polynomial in the first equation of (6.3.6).

Recall From Chapter 4 that a normal fan of a 2-dimensional convex polytope in $\mathbb{R}^2$ is the complete fan with apex at the origin, and 1-dimensional cones directed by the outward normal vectors of the 1-faces of this polytope. Recall that w_0, w_1 and w_2 do not belong to a line and denote by Δ the triangle with vertices w_0, w_1 and w_2. Let $\mathcal{E} \subset \mathbb{R}^2$ denote the normal fan of Δ. The triangle Δ together with $\mathcal{E}$ are represented in Figure 6.3 on the left. The 1-dimensional cones of $\mathcal{E}$ are $\mathsf{L}_0 = \{\lambda(0, -m_1) |\ \lambda \geq 0\}$, $\mathsf{L}_1 = \{\lambda(n_2, m_1 - m_2) |\ \lambda \geq 0\}$ and $\mathsf{L}_2 = \{\lambda(-n_2, m_2) |\ \lambda \geq 0\}$. Let C_0 (resp. C_1, C_2) denote the 2-dimensional cone generated by the two vectors $(0, -m_1)$ and $(-n_2, m_2)$ (resp. $(0, -m_1)$ and $(n_2, m_1 - m_2)$, $(n_2, m_1 - m_2)$ and $(-n_2, m_2)$), see Figure 6.3. In what follows, for $i = 0, 1, 2$, let $\mathring{\mathsf{C}}_i$ denote the relative interior of C_i and $\mathring{\mathsf{L}}_i$ denote the relative interior of L_i. The main result of this Section is the following one.

Theorem 6.15. *For $i = 0, 1, 2$, the set $\mathring{\mathsf{C}}_i$ cannot contain more than one tropical transversal intersection point of* (6.3.6). *Moreover, a 1-cone of $\mathcal{E}$ does not contain a transversal intersection point of T_1 and T_2. Finally, if T_1 and T_2 intersect non-transversally at a cell ξ, then ξ is contained in a 1-cone of the base fan $\mathcal{E}$.*

The proof of the first statement of this result is Corollary 6.23, and the proof of its second statement is Corollary 6.25. The last statement of Theorem 6.15 is proved by Lemma 6.26.

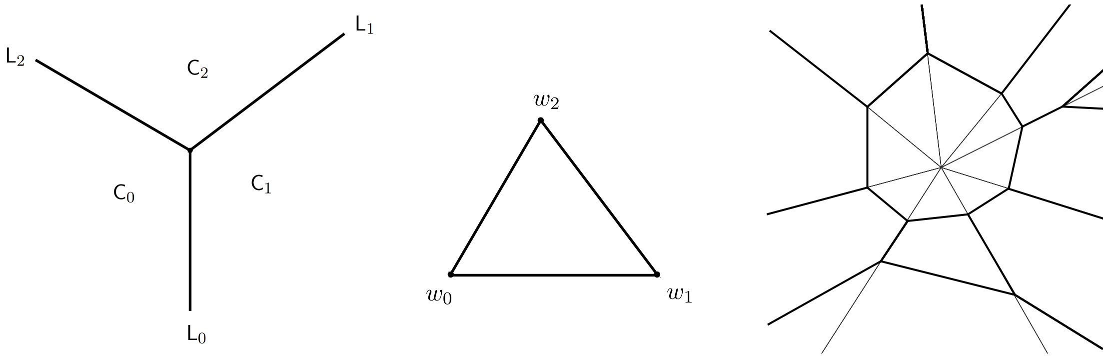

Figure 6.3: To the left: the base fan $\mathcal{E}$. To the right: a generic base fan.

Remark 6.16. *The 1-skeleton of the fan $\mathcal{E}$ is the tropical curve associated to $d_0 y^{w_0} + d_1 y^{w_1} + d_2 y^{w_2}$, for any $d_0, d_1, d_2 \in \mathbb{K}$ with zero valuation.*

Definition 6.17. *Let $\mathcal{C} \subset \mathbb{R}^2$ be a fan with 1-cones $\mathsf{J}_0, \mathsf{J}_1, \dots, \mathsf{J}_N$ and $T \subset \mathbb{R}^2$ be a tropical curve. We say that $\mathcal{C}$ is a **base fan** of T if for every vertex v of T, there exists a 1-cone J_i of $\mathcal{C}$ and a 1-face F of T adjacent to v such that $F \subset \mathsf{J}_i$.*

It is easy to check that if T has a base fan $\mathcal{C}$, then all of its vertices are located either on the 1-cones, or on the origin of $\mathcal{C}$ (see Figure 6.3 on the right for example). For obvious reasons, all results in this section on T_1 hold also true for the tropical curve T_2 associated to the polynomial appearing in the second equation of (6.3.6). Therefore, we state them only for T_1.

Lemma 6.18. *The fan $\mathcal{E}$ is a base fan of T_1.*

Proof. If $\alpha = 0$, then the result is immediate since the only vertex of T_1 is the center $(0, 0)$ of $\mathcal{E}$. Assume that $\alpha \neq 0$. Then, since (6.3.6) is highly non-degenerate, the subdivision τ_1 is a triangulation such that any triangle of τ_1 has at least two vertices in $\{w_0, w_1, w_2\}$. Assume without loss of generality that one such triangle is $[w_0, w_1, w_3]$. Therefore the edge $F_{0,1}$ dual to $[w_0, w_1]$ is adjacent to the vertex v_1 of T_1, dual to $[w_0, w_1, w_3] \subset \tau_1$. Note that

$$F_{0,1} = \{x \in \mathbb{R}^2 |\ \langle x, w_0 \rangle = \langle x, w_1 \rangle \geq \max(\langle x, w_2 \rangle,\ \langle x, w_3 \rangle - \alpha)\}. \tag{6.3.7}$$

It is clear that $F_{0,1}$ is contained in the line which contains the 1-cone L_0. We prove that $F_{0,1} \subset \mathsf{L}_0$ (see Fig. 6.4).

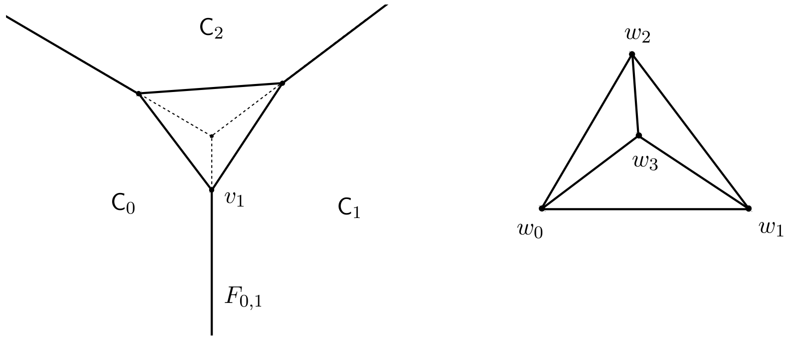

Figure 6.4: The edge $F_{0,1}$ is contained in L_0.

Assume that $F_{0,1}$ does not belong to L_0, we prove that this gives a contradiction. Consider a point $p \in F_{0,1} \setminus \mathsf{L}_0$. Therefore p is in $\mathring{\mathsf{C}}_2$. By Remark 6.16, we have $\langle p, w_2\rangle > \max\{\langle p, w_1\rangle, \langle p, w_0\rangle\}$ which is a contradiction to (6.3.7). □

Corollary 6.19. *Any vertex $v \neq (0,0)$ of T_1 contained in L_i for some $i \in \{0,1,2\}$, is 3-valent. Moreover, each 2-cone of $\mathcal{E}$ adjacent to L_i contains one edge of T_1 adjacent to v.*

Proof. Note that if T_1 has a vertex $v \neq (0,0)$, then $\alpha \neq 0$, and thus the 3-valency comes from the fact that τ_1 is a triangulation. Since $\mathcal{E}$ is a base fan of T_1, the second part of the corollary is a consequence of the balancing condition applied to v. □

Lemma 6.20. *Assume that T_1 has two vertices $v_i, v_j \neq (0,0)$ contained in distinct 1-cones L_i and L_j of $\mathcal{E}$, respectively. Then there exists an edge of T_1 that is adjacent to both v_i and v_j.*

Proof. Since both v_i and v_j are 3-valent vertices of T_1 (Corollary 6.19), their respective dual faces σ_i and σ_j are both triangles. The subdivision τ_1 cannot have more than three triangles since the support of the first equation of (6.3.6) has only four elements. Moreover, since Δ_1 is convex, any two triangles of τ_1 have one edge in common. Let $\delta_{i,j}$ denote the common edge of σ_i and σ_j. We have that the vertices v_i and v_j are joined by an edge of T_1, dual to $\delta_{i,j}$. □

Lemma 6.21. *T_1 cannot have more than one vertex on any 1-cone of $\mathcal{E}$.*

Proof. Consider a vertex $v \neq (0,0)$ of T_1 that belong to a 1-cone, say L_0. By Lemma 6.18, the vertex v is an endpoint of an edge $F_{0,1} \subset \mathsf{L}_0$ of T_1. Consequently

$$v \in \{x \in \mathbb{R}^2 |\ \langle x, w_0\rangle = \langle x, w_1\rangle \geq \max(\langle x, w_3\rangle - \alpha, \langle x, w_2\rangle)\}.$$

Note that for any x in L_0, we have $\langle x, w_0\rangle > \langle x, w_2\rangle$. Moreover, by Corollary 6.19, v is 3-valent, thus v is the unique point point $x \in \mathbb{R}^2$ such that $\langle x, w_0\rangle = \langle x, w_1\rangle = \langle x, w_3\rangle - \alpha > \langle x, w_2\rangle$. □

Lemma 6.22. *For $i = 0,1,2$, the set $\mathring{\mathsf{C}}_i$ cannot contain more than one edge of T_1.*

Proof. This is a consequence of Corollary 6.19, Lemma 6.20 and Lemma 6.21. □

Since Lemma 6.22 also applies on T_2, we have the following result.

Corollary 6.23. *A 2-cone of $\mathcal{E}$ contains at most one transversal intersection of T_1 and T_2.*

This proves the first statement of Theorem 6.15. To prove its second statement, we need the following Lemma.

Lemma 6.24. *If there exists an edge F of T_1 not contained in any 1-cone of $\mathcal{E}$ and intersecting one of these 1-cones in a point v, then v is an endpoint of F.*

Proof. Assume without loss of generality that $F \cap \mathsf{L}_0 \neq \emptyset$ and consider a point $v \in F \cap \mathsf{L}_0$. Since F is not contained in any 1-cone of $\mathcal{E}$, the relative interior of F is expressed as

$$\{x \in \mathbb{R}^2 \mid \langle x, w_i \rangle = \langle x, w_3 \rangle - \alpha > \max(\langle x, w_j \rangle, \langle x, w_k \rangle)\},$$

for distinct $i, j, k \in \{0, 1, 2\}$. Moreover, since $v \in \mathsf{L}_0$, we have

$$v \in \{x \in \mathbb{R}^2 \mid \langle v, w_0 \rangle = \langle v, w_1 \rangle\},$$

which means that v is not contained in the relative interior of F, and thus it is an endpoint of F. □

Corollary 6.25. *A 1-cone of $\mathcal{E}$ does not contain a transversal intersection point of T_1 and T_2.*

Proof. A transversal intersection point p of T_1 and T_2 is the intersection of the relative interior of an edge $F_1 \subset T_1$ and the relative interior of an edge $F_2 \subset T_2$. Lemma 6.24 shows that if there exists a point of L_0 belonging to the relative interiors of both F_1 and F_2, then both F_1 and F_2 are contained in L_0, which is impossible if the intersection is transversal. □

This finishes the proof of the second statement of Theorem 6.15. The following result finishes the proof of Theorem 6.15.

Recall that (6.4.1) is highly non-degenerate.

Lemma 6.26. *If T_1 and T_2 intersect non-transversally at a cell ξ, then ξ is contained in a 1-cone of the base fan $\mathcal{E}$.*

Proof. Assume that T_1 and T_2 intersect non-transversally at a cell ξ belonging to the relative interior of a 2-cone of $\mathcal{E}$, say of C_0, we prove that this gives a contradiction. We have that ξ is of type (I). Indeed, since $\mathcal{E}$ is a base fan of T_1 and of T_2, all vertices of the the latter tropical curves belong to the 1-cones of $\mathcal{E}$, and thus ξ cannot be of type (II), nor can it be of type (III). Therefore, ξ (which is of type (I)) is the intersection of the 1-dimensional cell $F_{0,3} \subset T_1$, dual to $[(0,0),(m_3,n_3)]$ and the 1-dimensional cell $F_{0,4} \subset T_2$, dual to $[(0,0),(m_4,n_4)]$. It follows that the points $(0,0)$, (m_3,n_3) and (m_4,n_4) belong to the same line, and thus (6.4.1) is not highly non-degenerate, a contradiction. □

Recall that we have $a_1 = b_1 = 1$.

Proposition 6.27. *Assume that T_1 and T_2 intersect transversally at a point $v \in \mathring{\mathsf{C}}_i, i \in \{0, 1, 2\}$. Then* $\operatorname{coef}(a_i)\operatorname{coef}(a_3) < 0$*,* $\operatorname{coef}(b_i)\operatorname{coef}(b_4) < 0$ *iff v is the valuation of a positive solution of* (6.3.6).

The proof of Proposition 6.27 follows from the next two Lemmas.

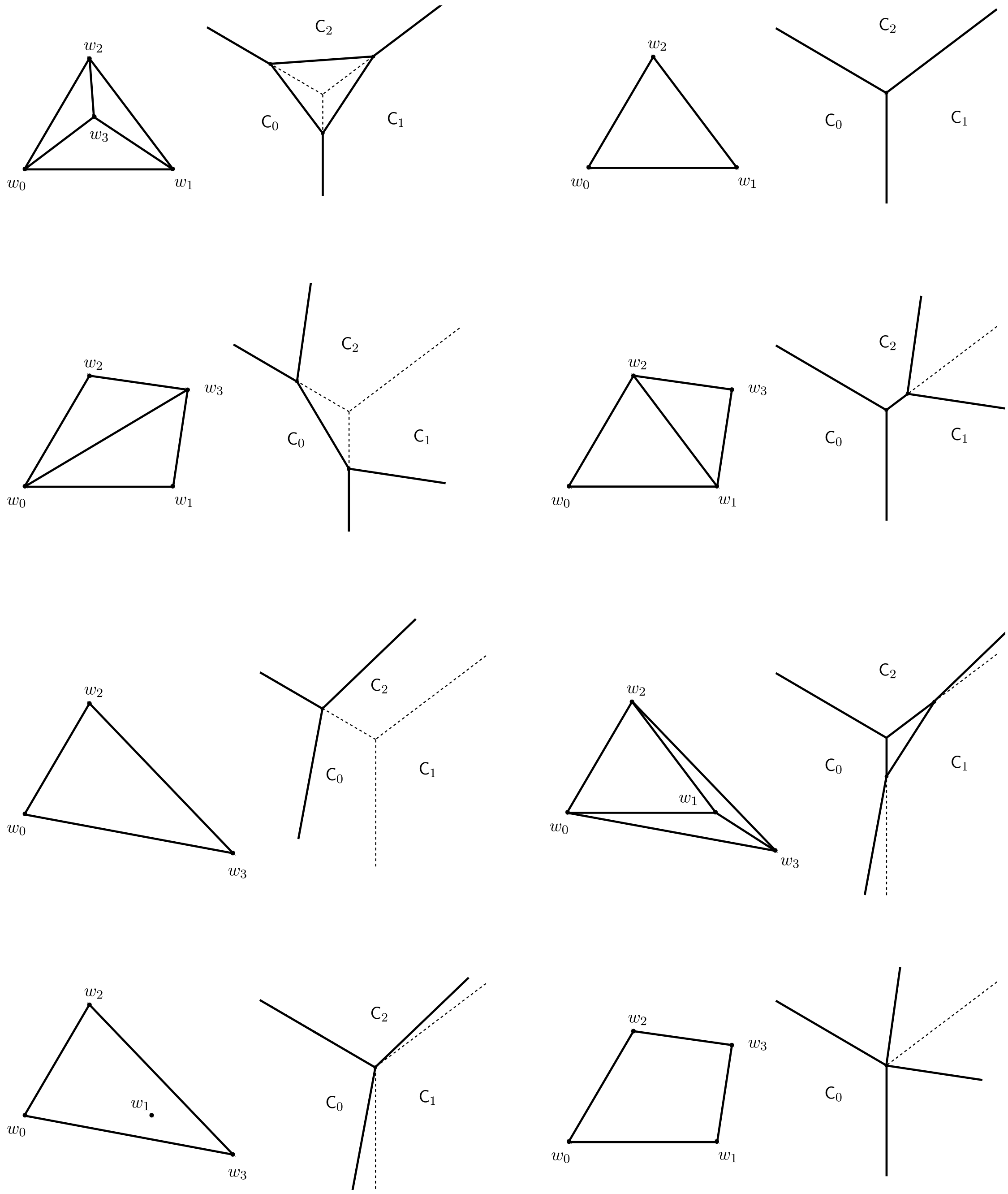

Figure 6.5: Disposition of T_1 with respect to its base fan $\mathcal{E}$ (together with its dual triangulation τ_1).

Lemma 6.28. *Let $v \in \mathring{\mathsf{C}}_i$ denote a transversal intersection point of T_1 and T_2. Then*

$$\langle v, w_i \rangle = \langle v, w_3 \rangle - \alpha = \langle v, w_4 \rangle - \beta > \max(\langle v, w_j \rangle, \langle v, w_k \rangle\},$$

satisfying that w_i, w_j and w_k are distinct points of $\{w_0, w_1, w_2\}$.

Proof. Assume without loss of generality that $v \in \mathring{\mathsf{C}}_0$, then $\langle v, w_0\rangle > \max(\langle v, w_1\rangle, \langle v, w_2\rangle)$. The proof comes directly from the fact that v belongs to the relative interior of an edge F_1 (resp. F_2) of T_1 (resp. T_2) defined by $\{x \in \mathbb{R}^2 |\ \langle x, w_0\rangle = \langle x, w_3\rangle - \alpha > \max(\langle x, w_1\rangle\ ,\ \langle x, w_2\rangle\}$ $\big($resp. $\{x \in \mathbb{R}^2 |\ \langle x, w_0\rangle = \langle x, w_4\rangle - \beta > \max(\langle x, w_1\rangle,\ \langle x, w_2\rangle\}\big)$. ☐

Lemma 6.29. *Assume that T_1 and T_2 intersect transversally at $v \in \mathring{\mathsf{C}}_i, i \in \{0, 1, 2\}$. Then the reduced system of* (6.3.6) *with respect to v is*

$$\operatorname{coef}(a_i)y^{w_i} + \operatorname{coef}(a_3)y^{w_3} = \operatorname{coef}(b_i)y^{w_i} + \operatorname{coef}(b_4)y^{w_4} = 0$$

Proof. Assume without loss of generality that $v := (v_1, v_2) \in \mathsf{C}_0$. Therefore, replacing (y_1, y_2) by $(t^{-v_1}y_1, t^{-v_2}y_2)$ in (6.3.1), we obtain

$$\begin{aligned} a_0t^{-\langle v,w_0\rangle}y^{w_0} + t^{-\langle v,w_1\rangle}y^{w_1} + a_2t^{-\langle v,w_2\rangle}y^{w_2} + a_3t^{\alpha-\langle v,w_3\rangle}y^{w_3} &= 0, \\ b_0t^{-\langle v,w_0\rangle}y^{w_0} + t^{-\langle v,w_1\rangle}y^{w_1} + b_2t^{-\langle v,w_2\rangle}y^{w_2} + b_4t^{\beta-\langle v,w_4\rangle}y^{w_4} &= 0. \end{aligned} \tag{6.3.8}$$

Using Lemma 6.28, the latter system can be expressed as

$$\begin{aligned} t^{-\langle v,w_0\rangle}\left(a_0y^{w_0} + t^{\langle v,w_0\rangle-\langle v,w_1\rangle}y^{w_1} + a_2t^{\langle v,w_0\rangle-\langle v,w_2\rangle}y^{w_2} + a_3y^{w_3}\right) &= 0, \\ t^{-\langle v,w_0\rangle}\left(b_0y^{w_0} + t^{\langle v,w_0\rangle-\langle v,w_1\rangle}y^{w_1} + b_2t^{\langle v,w_0\rangle-\langle v,w_2\rangle}y^{w_2} + b_4y^{w_4}\right) &= 0 \end{aligned} \tag{6.3.9}$$

where each of $\langle v, w_0\rangle - \langle v, w_1\rangle$ and $\langle v, w_0\rangle - \langle v, w_2\rangle$ are positive. Therefore, for $t > 0$ small enough, the system (6.3.9) becomes

$$\operatorname{coef}(a_0)y^{w_0} + \operatorname{coef}(a_3)y^{w_3} = \operatorname{coef}(b_0)y^{w_0} + \operatorname{coef}(b_4)y^{w_4} = 0.$$

☐

6.4 Preliminary case-by-case analysis for $n = k = 2$

Recall that a system of type $n = k = 2$ is said to be highly non-degenerate if no three points of its support belong to a line. Furthermore, recall that a normalized system is of the form

$$\begin{aligned} a_0 + y_1^{m_1} + a_2y_1^{m_2}y_2^{n_2} + a_3t^{\alpha}y_1^{m_3}y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2y_1^{m_2}y_2^{n_2} + b_4t^{\beta}y_1^{m_4}y_2^{n_4} &= 0. \end{aligned} \tag{6.4.1}$$

satisfying that all a_i and b_j are in $\mathbb{RK}^*$ and verify $\operatorname{ord}(a_i) = \operatorname{ord}(b_j) = 0$, all w_i are in $\mathbb{Z}^2$, both m_1, n_2 are positive and both α, β are real numbers.

Consider a highly non-degenerate, normalized system (6.4.1).

Lemma 6.30. *Assume that the system* (6.4.1)*, satisfies one of the following:* $\operatorname{coef}(a_0) = \operatorname{coef}(b_0)$, $\operatorname{coef}(a_2) = \operatorname{coef}(b_2)$ *or* $\operatorname{coef}(a_0)/\operatorname{coef}(a_2) \neq \operatorname{coef}(b_0)/\operatorname{coef}(b_2)$*. Then, one can associate to* (6.4.1) *a highly non-degenerate normalized system*

$$\begin{aligned} c_0 + z_1^{\hat{m}_1} + c_2z_1^{\hat{m}_2}z_2^{\hat{n}_2} + c_3t^{\gamma}z_1^{\hat{m}_3}z_2^{\hat{n}_3} &= 0, \\ d_0 + z_1^{\hat{m}_1} + d_2z_1^{\hat{m}_2}z_2^{\hat{n}_2} + d_4t^{\delta}z_1^{\hat{m}_4}z_2^{\hat{n}_4} &= 0 \end{aligned} \tag{6.4.2}$$

with equations in $\mathbb{RK}[z_1^{\pm1}, z_2^{\pm1}]$ that has the same number of non-degenerate positive solutions as (6.4.1)*, where* $\operatorname{coef}(c_i) = \operatorname{coef}(d_i)$ *for* $i = 0, 2$.

Proof. First, the result becomes trivial if (6.4.1) satisfies at least two of the equalities of the Lemma. Indeed, then the three equalities will hold automatically and thus it suffices to consider (6.4.1) itself, and thus proving the result. Therefore, we assume that only one of the mentioned equalities holds true.

• Assume that $\mathrm{coef}(a_0) = \mathrm{coef}(b_0)$. The system

$$\begin{array}{rcl} \tilde{a}_0 t^{\alpha_0} + (b_2 - a_2) y_1^{m_2} y_2^{n_2} - a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} & = & 0, \\ \mathrm{coef}(a_0)\left(\tilde{a}_1 t^{\alpha_1} y_1^{m_1} + \tilde{b}_2 y_1^{m_2} y_2^{n_2} - \tilde{a}_3 t^{\alpha} y_1^{m_3} y_2^{n_3} + \tilde{b}_4 t^{\beta} y_1^{m_4} y_2^{n_4}\right) & = & 0, \end{array} \tag{6.4.3}$$

has the same number of non-degenerate positive solutions of (6.4.1). Indeed, the first equation of (6.4.3) is obtained by subtracting the first equation of (6.4.1) from its second one, whereas the second equation of (6.4.3) is obtained by multiplying the second equation of (6.4.1) by $\mathrm{coef}(a_0)/b_0$ and subtracting from it the first equation of (6.4.1) multiplied by $\mathrm{coef}(a_0)/a_0$. Note that $\tilde{a}_1 t^{\alpha_1} = b_0^{-1} - a_0^{-1}$, $\tilde{a}_3 = (a_3/a_0)$, $\tilde{b}_2 = (b_2/b_0 - a_2/a_0)$, $\tilde{b}_4 = (b_4/b_0)$, $\tilde{a}_0 t^{\alpha_0} = b_0 - a_0$, $\mathrm{ord}(\tilde{a}_i) = \mathrm{ord}(\tilde{b}_j) = 0$, $\alpha_i > 0$ for $i = 0, 1$. Moreover, since $\mathrm{coef}(a_0) = \mathrm{coef}(b_0)$, we have

$$\mathrm{coef}\left(\frac{\mathrm{coef}(a_0)a_3}{a_0}\right) = \mathrm{coef}(a_3), \quad \mathrm{coef}\left(\frac{\mathrm{coef}(a_0)b_4}{b_0}\right) = \mathrm{coef}(b_4)$$

and

$$\mathrm{coef}\left(\mathrm{coef}(a_0)\left(\frac{b_2}{b_0} - \frac{a_2}{a_0}\right)\right) = \mathrm{coef}(b_2 - a_2).$$

Since we are only interested in positive solutions of (6.4.3), *dividing* the first and the second equation of (6.4.3) by $-a_3 y_1^{m_2} y_2^{n_2}$ and $-(\mathrm{coef}(a_0)a_3/a_0) y_1^{m_2} y_2^{n_2}$ respectively will not change the number of non-degenerate positive solutions of (6.4.3). Moreover, this number of non-degenerate positive solutions will not change if we replace (y_1, y_2) by $(t^k y_1, t^l y_2)$ in (6.4.3) for some real numbers k and l satisfying $\langle (k,l), (m_3 - m_2, n_3 - n_2)\rangle - \alpha = \langle (k,l), (m_4 - m_2, n_4 - n_2)\rangle - \beta = 0$. The system we obtain is

$$\begin{array}{rcccccccl} c_3 t^{\gamma} y_1^{-m_2} y_2^{-n_2} & + & c_0 & + & y_1^{m_3-m_2} y_2^{n_3-n_2} & + & c_2 y_1^{m_4-m_2} y_2^{n_4-n_2} & = & 0, \\ d_4 t^{\delta} y_1^{m_1-m_2} y_2^{-n_2} & + & d_0 & + & y_1^{m_3-m_2} y_2^{n_3-n_2} & + & d_2 y_1^{m_4-m_2} y_2^{n_4-n_2} & = & 0, \end{array} \tag{6.4.4}$$

with

$$c_0 = -\frac{b_2 - a_2}{a_3}, \quad c_2 = -\frac{b_4}{a_3}, \quad c_3 = -\frac{\tilde{a}_0}{a_3}, \quad d_0 = -\frac{a_0}{a_3}\left(\frac{b_2}{b_0} - \frac{a_2}{a_0}\right), \quad d_2 = -\frac{b_4 a_0}{a_3 b_0},$$

$$d_4 = -\frac{a_0}{a_3}, \quad \gamma = \alpha_0 + \langle (-m_2, -n_2), (k,l)\rangle \quad \text{and} \quad \delta = \alpha_1 + \langle (m_1 - m_2, -n_2), (k,l)\rangle.$$

From $\mathrm{coef}(a_0) = \mathrm{coef}(b_0)$ and

$$\mathrm{coef}\left(\mathrm{coef}(a_0)\left(\frac{b_2}{b_0} - \frac{a_2}{a_0}\right)\right) = \mathrm{coef}(b_2 - a_2),$$

we have $\mathrm{coef}(c_2) = \mathrm{coef}(d_2)$ and $\mathrm{coef}(c_0) = \mathrm{coef}(d_0)$. Moreover, all $\mathrm{ord}(c_i)$ and $\mathrm{ord}(\hat{b}_j)$ are zero.

We make the monomial change of coordinates $(y_1, y_2) \mapsto (z_1, z_2)$ of $(\mathbb{RK}^*)^2$ such that $y_1^{m_3-m_2} y_2^{n_3-n_2} = z_1^{\hat{m}_1}$ and $y_1^{m_4-m_2} y_2^{n_4-n_2} = z_1^{\hat{m}_2} z_2^{\hat{n}_2}$, where both $\hat{m}_1$ and $\hat{n}_2$ are integers. Finally, replacing z_1 (resp. z_2) by z_1^{-1} (resp. z_2^{-1}) if necessary (since the solutions that we are interested in are non-zero), we have $\hat{m}_1, \hat{n}_2 > 0$, and thus we obtain a highly non-degenerate normalized system (6.4.2) satisfying the conditions of the Lemma.

• Assume that $\frac{\operatorname{coef}(a_0)}{\operatorname{coef}(a_2)} = \frac{\operatorname{coef}(b_0)}{\operatorname{coef}(b_2)}$. Dividing the first (resp. second) equation of (6.4.1) by a_2 (resp. b_2), and making the monomial change of coordinates $(y_1, y_2) \mapsto (z_1, z_2)$ such that $z_1^{\tilde{m}_1} = y_1^{m_2} y_2^{n_2}$ and $z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} = y_1^{m_1}$. Thus we obtain the highly non-degenerate system

$$\begin{aligned} a_0/a_2 + z_1^{\tilde{m}_1} + (1/a_2) z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} + (a_3/a_2) t^{\alpha} z_1^{\tilde{m}_3} z_2^{\tilde{n}_3} &= 0, \\ b_0/b_2 + z_1^{\tilde{m}_1} + (1/b_2) z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} + (b_4/b_2) t^{\beta} z_1^{\tilde{m}_4} z_2^{\tilde{n}_4} &= 0. \end{aligned} \tag{6.4.5}$$

Since we are interested in non-zero solutions, replacing z_1, z_2 by z_1^{-1}, z_2^{-1} if necessary, we assume that both $\tilde{m}_1$ and $\tilde{n}_2$ are positive. Therefore, the system (6.4.5) is a normalized system with $\operatorname{coef}(a_0/a_2) = \operatorname{coef}(b_0/b_2)$ and $\operatorname{coef}(1/a_2) \neq \operatorname{coef}(1/b_2)$. Note that such a change of coordinates does not change the number of non-degenerate positive solutions. Applying the proof of the case of $\operatorname{coef}(a_0) = \operatorname{coef}(b_0)$ to (6.4.5) gives the result.

• Assume that $\operatorname{coef}(a_2) = \operatorname{coef}(b_2)$. Similarly to the case where $\dfrac{\operatorname{coef}(a_0)}{\operatorname{coef}(a_2)} = \dfrac{\operatorname{coef}(b_0)}{\operatorname{coef}(b_2)}$, we make coordinate changes and monomial divisions on (6.4.1) to reduce to the already proven case where $\operatorname{coef}(a_0) = \operatorname{coef}(b_0)$.

□

Lemma 6.31. *Assume that the coefficients of the system* (6.4.1) *satisfy* $\operatorname{coef}(a_i) \neq \operatorname{coef}(b_i)$ *for* $i = 0, 2$, $\operatorname{coef}(a_0)/\operatorname{coef}(a_2) \neq \operatorname{coef}(b_0)/\operatorname{coef}(b_2)$ *and* $\alpha\beta = 0$. *Then one can associate to* (6.4.1) *a highly non-degenerate normalized system*

$$\begin{aligned} c_0 + z_1^{\tilde{m}_1} + c_2 z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} + c_3 t^{\gamma} z_1^{\tilde{m}_3} z_2^{\tilde{n}_3} &= 0, \\ d_0 + z_1^{\tilde{m}_1} + d_2 z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} + d_4 t^{\delta} z_1^{\tilde{m}_4} z_2^{\tilde{n}_4} &= 0 \end{aligned} \tag{6.4.6}$$

with equations in $\mathbb{RK}[z_1^{\pm 1}, z_2^{\pm 1}]$ *that has the same number of non-degenerate positive solutions as* (6.4.1), *where* $\operatorname{coef}(c_i) \neq \operatorname{coef}(d_i)$ *for* $i = 0, 2$,
$\operatorname{coef}(c_0)/\operatorname{coef}(c_2) \neq \operatorname{coef}(d_0)/\operatorname{coef}(d_2)$ *and* $\gamma\delta \neq 0$.

Proof. Assume the hypotheses of the Lemma on (6.4.1), and assume without loss of generality that only α is equal to zero. Replace (y_1, y_2) by $(t^k y_1, t^l y_2)$ in (6.4.1) so that $\langle (k,l), (m_2, n_2) \rangle = 0$ and $\langle (k,l), (m_4, n_4) \rangle = -\beta$. Since (6.4.1) is highly non-degenerate, we have $\langle (k,l), (m_1, 0) \rangle = \gamma_1 \neq 0$ and $\langle (k,l), (m_3, n_3) \rangle = \gamma_3 \neq 0$. The system

$$\begin{aligned} b_0/b_4 + y_1^{m_4} y_2^{n_4} + (b_2/b_4) y_1^{m_2} y_2^{n_2} + (1/b_4) t^{\gamma_1} y_1^{m_1} &= 0, \\ (b_0 - a_0)/b_4 + y_1^{m_4} y_2^{n_4} + ((b_2 - a_2)/b_4) y_1^{m_2} y_2^{n_2} - (a_3/b_4) t^{\gamma_3} y_1^{m_3} y_2^{n_3} &= 0 \end{aligned}$$

has the same number of non-degenerate positive solutions as (6.4.1). Indeed, the second equation of the latter system is obtained by subtracting the first equation of (6.4.1) divided by b_4 from its second one also divided by b_4. Doing a monomial change of variables $(y_1, y_2) \mapsto (z_1, z_2)$ so that $z_1^{\tilde{m}_1} = y_1^{m_4} y_2^{n_4}$ and $z_1^{\tilde{m}_2} z_2^{\tilde{n}_2} = y_1^{m_2} y_2^{n_2}$ satisfying $\tilde{m}_1 > 0$ and $\tilde{n}_2 > 0$. The result comes from deducing that $b_0/b_2 \neq (b_0 - a_0)/(b_2 - a_2)$.

□

Remark 6.32. *Thanks to Lemmata 6.30 and 6.31, we only need to consider the following two cases.*

$$(\alpha,\beta)\neq(0,0)\quad \textit{and}\quad \operatorname{coef}(a_i)=\operatorname{coef}(b_i)\quad \textit{for}\quad i=0,2 \tag{6.4.7}$$

and

$$\alpha\beta\neq 0,\ \operatorname{coef}(a_i)\neq\operatorname{coef}(b_i)\quad \textit{for}\quad i=0,2\quad \textit{and}\quad \frac{\operatorname{coef}(a_0)}{\operatorname{coef}(a_2)}\neq\frac{\operatorname{coef}(b_0)}{\operatorname{coef}(b_2)}. \tag{6.4.8}$$

We start this Section (see Subsection 6.4.1), by writing explicitly approximation polynomials of (6.4.1) for some cells of type (I). The remaining part is mainly devoted to explicitly writing the reduced systems of (6.4.1) with respect to non-transversal intersection points of type (II) and (III). We also give some key results that we will frequently refer to in the rest of this chapter.

Let Δ_1 and Δ_2 (resp. τ_1 and τ_2, T_1 and T_2) denote the Newton polytopes (resp. dual subdivisions, tropical curves) associated to the first and second equation of (6.4.1) respectively.

It will be useful for the computations in the following sections to write explicitly the coordinates of vertices of each of T_1 and T_2. Recall that if T_1 (resp. T_2) has a vertex v_1 (resp. v_2) that belongs to the relative interior of a 1-cone of $\mathcal{E}$, then it is dual to the triangle $\Delta_{v_1}\in\tau_1$ (resp. $\Delta_{v_2}\in\tau_2$) with vertices (m_i,n_i), (m_j,n_j) and (m_3,n_3) (resp. (m_4,n_4)) for distinct $i,j\in\{0,1,2\}$.

For obvious reasons, the following coordinates of the possible vertices of T_1 also hold true for the possible vertices of T_2 by replacing (m_3,n_3) and α by (m_4,n_4) and β. Therefore, we state them only for T_1 and distinguish three cases.

- First case: $v_1\in\mathsf{L}_0$. The coordinates (x_1,x_2) of v_1 satisfy $0=m_1x_1=m_3x_1+n_3x_2-\alpha$, and thus $(x_1,x_2)=(0,\alpha/n_3)$.
- Second case: $v_1\in\mathsf{L}_1$. The coordinates (x_1,x_2) of v_1 satisfy $m_1x_1=m_2x_1+n_2x_2=m_3x_1+n_3x_2-\alpha$, and thus
$$(x_1,x_2)=\left(\frac{n_2\alpha}{(m_3-m_1)n_2-(m_2-m_1)n_3}\quad,\quad -\frac{(m_2-m_1)\alpha}{(m_3-m_1)n_2-(m_2-m_1)n_3}\right).$$
- Third case: $v_1\in\mathsf{L}_2$. The coordinates (x_1,x_2) of v_1 satisfy $0=m_2x_1+n_2x_2=m_3x_1+n_3x_2-\alpha$, and thus
$$(x_1,x_2)=\left(\frac{n_2\alpha}{m_3n_2-m_2n_3}\quad,\quad -\frac{m_2\alpha}{m_3n_2-m_2n_3}\right).$$

6.4.1 Approximation polynomials for type-(I) intersections

In this subsection, we assume that T_1 and T_2 intersect non-transversally at distinct cells $\mathfrak{E}_i\subset\mathsf{L}_i$ and $\mathfrak{E}_j\subset\mathsf{L}_j$ for $i,j\in\{0,1,2\}$, both of type (I), and that each of $\mathring{\mathfrak{E}}_i$ and $\mathring{\mathfrak{E}}_j$ contains the valuations of non-degenerate positive solutions of (6.4.1). Then, we have the following result.

Lemma 6.33. *If T_1 and T_2 intersect non-transversally at a cell $\mathfrak{E}_k\subset\mathsf{L}_k$ of type (I), different from $\mathfrak{E}_i$ and from $\mathfrak{E}_j$, then $\mathring{\mathfrak{E}}_k$ does not contain the valuation of any non-degenerate positive solution of* (6.4.1).

We may assume without loss of generality that $i = 0$ and $j = 2$, and thus $k = 1$.

Proof of Lemma 6.33. Assume that T_1 and T_2 intersect non-transversally at a cell $\mathfrak{E}_1 \subset \mathsf{L}_1$ of type (I). Since each of $\mathring{\mathfrak{E}}_0$ and $\mathring{\mathfrak{E}}_2$ contains the valuations of non-degenerate positive solutions of (6.4.1), using same arguments as in the proof of Proposition 6.6, we have $\operatorname{coef}(a_0)\operatorname{coef}(a_2) < 0$ (resp. $\operatorname{coef}(b_0)\operatorname{coef}(b_2) < 0$) and $\operatorname{coef}(a_0) < 0$ (resp. $\operatorname{coef}(b_0) < 0$). Therefore $\operatorname{coef}(a_2) > 0$ and $\operatorname{coef}(b_2) > 0$, and consequently, the reduced system $y_1^{m_1} + \operatorname{coef}(a_2)y_1^{m_2}y_2^{n_2} = y_1^{m_1} + \operatorname{coef}(b_2)y_1^{m_2}y_2^{n_2} = 0$, associated to $\mathfrak{E}_1$, does not have positive solutions. □

We want to find an approximation polynomial for each of $\mathfrak{E}_0$ and $\mathfrak{E}_2$. Consider the following polynomials

$$f_{0,t} = \operatorname{coef}(c_0)t^{\gamma_0} + \operatorname{coef}(c_2)t^{\gamma_2}y^{n_2} - \operatorname{coef}(a_3)t^{\alpha}y^{n_3} + \operatorname{coef}(b_4)t^{\beta}y^{n_4} \tag{6.4.9}$$

and

$$f_{2,t} = ct^{\delta} - \operatorname{coef}(a_3)t^{\alpha}y^{\frac{m_3n_2-m_2n_3}{n_2}} + \operatorname{coef}(b_4)t^{\beta}y^{\frac{m_4n_2-m_2n_4}{n_2}}, \tag{6.4.10}$$

with $c_i = b_i - a_i$, $\gamma_i = \operatorname{ord}(c_i)$ for $i = 0, 2$ and ct^{δ} is the first-order term of $c_2 - c_0$.

Lemma 6.34. *The polynomials $f_{0,t}$ and $f_{2,t}$ are approximation polynomials of* (6.4.1) *for $\mathfrak{E}_0$ and $\mathfrak{E}_2$ respectively.*

Proof. Since $\mathring{\mathfrak{E}}_0$ and $\mathring{\mathfrak{E}}_2$ both contain valuations of non-degenerate positive solutions of (6.4.1), using arguments similar to those appearing in the proof of Proposition 6.6, we may assume without loss of generality that $\operatorname{coef}(a_0) = \operatorname{coef}(b_0) = -1$ and $\operatorname{coef}(a_2) = \operatorname{coef}(b_2) = 1$.

The system (6.4.1) already satisfies all properties of Proposition 6.6, in particular, the cell $\mathring{\mathfrak{E}}_0 \subset \mathsf{L}_0$ is contained in $\{0\}\times\,]-\infty, 0[$. Therefore, the fact that $f_{0,t}$ is an approximation polynomial of (6.4.1) for $\mathfrak{E}_0$ is straightforward.

A non-degenerate positive solution $(\nu, \varrho) \in (\mathbb{RK}^*)^2$ of (6.4.1) with valuation in $\mathring{\mathfrak{E}}_2$ satisfies $\operatorname{coef}(\nu)^{m_2}\operatorname{coef}(\varrho)^{n_2} - 1 = 0$. Indeed, $y_1^{m_2}y_2^{n_2} - 1 = 0$ is the reduced system associated to $\mathfrak{E}_2$. Therefore, $\nu^{m_2}\varrho^{n_2} = 1 + \mu$ with $\mu \in \mathbb{RK}$ and $\operatorname{ord}(\mu) > 0$, thus $\varrho = \nu^{-\frac{m_2}{n_2}}(1+\mu)^{\frac{1}{n_2}}$. We have that the system

$$\begin{aligned} a_0 + z_2^{m_1} + a_2z_1 + a_3t^{\alpha}z_1^{\frac{m_3}{n_2}}z_2^{\frac{m_3n_2-m_2n_3}{n_2}} &= 0, \\ b_0 + z_2^{m_1} + b_2z_1 + b_4t^{\beta}z_1^{\frac{m_4}{n_2}}z_2^{\frac{m_4n_2-m_2n_4}{n_2}} &= 0, \end{aligned} \tag{6.4.11}$$

obtained via the monomial change of coordinates $(y_1, y_2) \to (z_1, z_2)$ defined by $z_1 = y_1^{m_2}y_2^{n_2}$ and $z_2 = y_1$, has the same number of non-degenerate solutions in $(\mathbb{RK}^*)^2$ as (6.4.1). We now prove that (6.4.11) satisfies all the properties of Proposition 6.6. Similarly to the proof of Proposition 6.6, we deduce from the latter change of coordinates that the tropical curves of the system (6.4.11) intersect non-transversally at a cell $\tilde{\mathfrak{E}}_2$ of type (I). Moreover, the systems (6.4.11) and (6.4.1) have the same number of non-degenerate positive solutions with valuations in $\mathring{\tilde{\mathfrak{E}}}_2$ and $\mathring{\mathfrak{E}}_2$ respectively. This proves that (6.4.11) satisfies property **ii)** of Proposition 6.6.

We have that (x_1, x_2) belongs to $\mathring{\tilde{\mathfrak{E}}}_2$ if and only if it satisfies

$$0 = x_1 > \max\{m_1x_2\ , -\alpha + m_4x_1/n_2 + (m_4n_2 - m_2n_4)x_2/n_2\}$$

and

$$0 = x_1 > \max\{m_1x_2\ , -\alpha + m_3x_1/n_2 + (m_3n_2 - m_2n_3)x_2/n_2\}.$$

Therefore, since $m_1 > 0$ and $m_1x_2 < 0$ for $(x_1, x_2) \in \mathring{\tilde{\mathfrak{E}}}_2$, we have $\mathring{\tilde{\mathfrak{E}}}_2 \subset \{0\}\times\,]-\infty, 0[$. Moreover, from $\mathrm{coef}(a_0) = \mathrm{coef}(b_0) = -1$ and $\mathrm{coef}(a_2) = \mathrm{coef}(b_2) = 1$, we deduce that (6.4.11) satisfies property **i)** of Proposition 6.6. Therefore $f_{2,t}$ is an approximation polynomial of (6.4.1) for $\tilde{\mathfrak{E}}_2$. □

In Sections 6.6, 6.7 and 6.5, we use $f_{0,t}$ and $f_{2,t}$ of (6.4.9) and (6.4.10) respectively, to compute the non-degenerate positive solutions of (6.4.1) with valuations in $\mathring{\mathfrak{E}}_0$ and $\mathring{\mathfrak{E}}_2$ respectively.

Remark 6.35. *By Descartes' rule of sign applied to $f_{0,t}$ (resp. $f_{2,t}$), the cell $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$) contains the valuations of at most three (resp. two) positive solutions of* (6.4.1).

In what follows, we denote by Γ_0 and Γ_2 the lower hulls associated to $f_{0,t}$ and $f_{2,t}$ respectively (see Figure 6.20 for example).

Remark 6.36. *If v is a vertex of Γ_0, then v belongs to the set*

$$\{(0, \gamma_0),\ (n_2, \gamma_2),\ (n_3, \alpha),\ (n_4, \beta)\} \subset \mathbb{R}^2.$$

Similarly, if v is a vertex of Γ_2, then v belongs to the set

$$\{(0, \delta),\ (\frac{m_3n_2 - m_2n_3}{n_2}, \alpha),\ (\frac{m_4n_2 - m_2n_4}{n_2}, \beta)\}.$$

Definition 6.37. *We say that Γ_0 (resp. Γ_2) is **optimally sloped** if it does not have edges with positive slope and it contains all the points of the set $\{(0, \gamma_0),\ (n_2, \gamma_2),\ (n_3, \alpha),\ (n_4, \beta)\}$ (resp. $\{(0, \delta),\ (\frac{m_3n_2-m_2n_3}{n_2}, \alpha),\ (\frac{m_4n_2-m_2n_4}{n_2}, \beta)\}$).*

Example 6.38. Consider the particular system (6.4.1)

$$\begin{aligned} -1 + t^{12} + x^6 + x^3y^6 - tx^{10}y^{12} &= 0, \\ -1 + x^6 + (1 + t^5)x^3y^6 - t^{1.5}x^7y^{11} &= 0. \end{aligned} \tag{6.4.12}$$

The corresponding approximation polynomials (6.4.9) and (6.4.10) are $f_{0,t}(y) = -t^{12} + t^5y^6 - t^{1.5}y^{11} + ty^{12}$ and $f_{2,t}(y) = t^5 + ty^4 - t^{1.5}y^{\frac{5}{3}}$. Applying Corollary 6.12, we have that if (6.4.12) has six positive solutions, then the first terms of the positive solutions of (6.4.12) with valuations in the relative interior $\mathring{\mathfrak{E}}_0$ of the cell $\mathfrak{E}_0$ are $\left(1,\ t^{\frac{1}{2}}\right)$, $\left(1,\ t^{\frac{7}{10}}\right)$ and $\left(1,\ t^{\frac{7}{6}}\right)$, and those with valuations in $\mathring{\tilde{\mathfrak{E}}}_2$ are $\left(t^{\frac{1}{30}}c_1, t^{-\frac{1}{60}}\sqrt{c_1}\right)$ and $\left(t^{\frac{7}{45}}c_2, t^{-\frac{7}{90}}\sqrt{c_2}\right)$ for some $c_1, c_2 \in \mathbb{R}^*$.

The valuations of these solutions are represented in Figure 6.6. Note that this system (6.4.12) has also a non-degenerate positive solution with valuation a transversal intersection point $\left(-\frac{4}{11}, \frac{13}{22}\right)$. The system

$$\begin{aligned} -1 + t^{12} + x^6 + x^3y^6 - tx^{10}y^{12} &= 0, \\ -t^{12} + t^5x^3y^6 - t^{1.5}x^7y^{11} + tx^{10}y^{12} &= 0 \end{aligned} \tag{6.4.13}$$

has the same non-degenerate positive solutions as (6.4.12). Indeed, the second equation of (6.4.13) is obtained by subtracting the second equation of (6.4.12) from its first one. The tropical curves associated to (6.4.13) intersect transversally in six points (see Fig 6.7). This shows that, since in this case the curves T_1 and T_2 intersect transversally, the bound six of Lemma 6.4 is sharp.

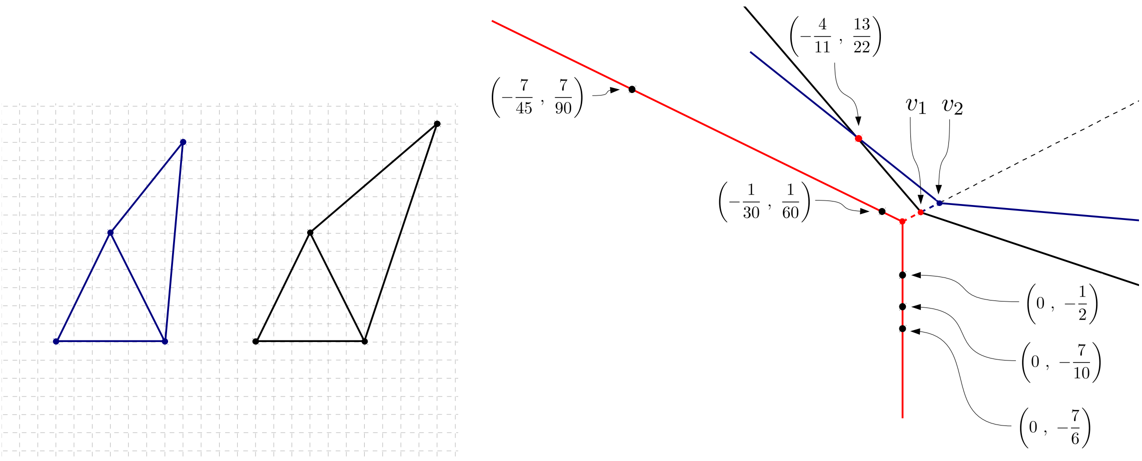

Figure 6.6: Five solutions of (6.4.12) with valuations contained in cells of type (I).

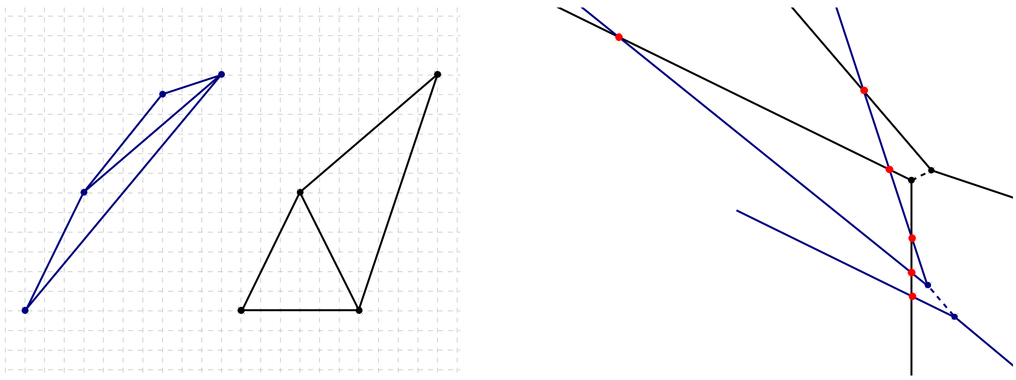

Figure 6.7: Tropical curves of a system of type $n = k = 2$ intersect transversally at six points.

6.4.2 Reduced systems for type-(II) intersections

We start with the following result.

Lemma 6.39. *If T_1 and T_2 intersect non-transversally at a point v of type (II), then T_1 and T_2 intersect non-transversally at a cell of type (I) such that v is one of its endpoints.*

Proof. Assume that T_1 and T_2 intersect non-transversally at a point v of type (II). Then by Lemma 6.26, the point v belongs to one of the 1-cones of $\mathcal{E}$, say L_0. By definition, the point v is the intersection of a vertex v_1 of T_1 and the relative interior of a facet F_2 of T_2. By Lemma 6.24, we have $F_2 \subset \mathsf{L}_0$. Since $\mathcal{E}$ is a base fan of T_1, the latter tropical curve has a facet $F_1 \subset \mathsf{L}_0$ adjacent to v_1, and thus $F_1 \cap F_2$ is of type (I) and v is an endpoint of $F_1 \cap F_2$. □

Corollary 6.40. *The reduced system with respect to a non-transversal intersection point of type (II) is of the form*

$$\mathrm{coef}(a_i)y_1^{m_i}y_2^{n_i} + \mathrm{coef}(a_j)y_1^{m_j}y_2^{n_j} = \mathrm{coef}(b_i)y_1^{m_i}y_2^{n_i} + \mathrm{coef}(b_j)y_1^{m_j}y_2^{n_j} + \mathrm{coef}(b_4)y_1^{m_4}y_2^{n_4} = 0$$

or

$$\mathrm{coef}(b_i)y_1^{m_i}y_2^{n_i} + \mathrm{coef}(b_j)y_1^{m_j}y_2^{n_j} = \mathrm{coef}(a_i)y_1^{m_i}y_2^{n_i} + \mathrm{coef}(a_j)y_1^{m_j}y_2^{n_j} + \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} = 0,$$

for some distinct $i, j \in \{0, 1, 2\}$.

Remark 6.41. *Each system appearing in Corollary 6.40 is composed of two equations in two variables and having a total of three distinct monomials. Therefore, the reduced system with valuation a non-transversal intersection point of type (II) has at most one positive solution.*

6.4.3 Reduced systems for type-(III) intersections at the origin

The tropical curves T_1 and T_2 intersect non-transversally at a point v_0 of type (III) that is the origin of $\mathcal{E}$ if and only if $\alpha, \beta \geq 0$. In this Subsection, we assume $0 \leq \alpha \leq \beta$ and $(\alpha, \beta) \neq (0, 0)$.

The system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ c_0 t^{\gamma_0} + c_2 t^{\gamma_2} y_1^{m_2} y_2^{n_2} - a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0, \end{aligned} \tag{6.4.14}$$

with $c_i t^{\gamma_i} = b_i - a_i$, $\mathrm{ord}(c_i) = 0$ and $\gamma_i \geq 0$ for $i = 0, 2$, has the same number of non-degenerate positive solutions as (6.4.1). Indeed, the second equation of (6.4.14) is obtained by substracting the first equation of (6.4.1) from its second one.

If $\mathrm{coef}(a_i) \neq \mathrm{coef}(b_i)$ for $i = 0, 2$, and $\alpha\beta \neq 0$, then $\gamma_0 = \gamma_2 = 0$ and the reduced system

$$\mathrm{coef}(a_0) + y_1^{m_1} + \mathrm{coef}(a_2)y_1^{m_2}y_2^{n_2} = \mathrm{coef}(c_0) + \mathrm{coef}(c_2)y_1^{m_2}y_2^{n_2} = 0$$

with respect to v_0 has at most one positive solution (the case of a simplex).

Assume now that $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ for $i = 1, 2$. Then $\gamma_0, \gamma_2 > 0$, and we distinguish the following cases.

i) First case: there exists only one element of the set $\{\alpha, \beta, \gamma_0, \gamma_2\}$ that is equal to $\min(\alpha, \beta, \gamma_0, \gamma_2)$. The reduced system of (6.4.14) with respect to v_0 has no positive solutions.

ii) Second case: $\gamma_0 = \gamma_2 < \min(\alpha, \beta)$. Then the reduced system of (6.4.14) with respect to v_0 becomes

$$\mathrm{coef}(a_0) + y_1^{w_1} + \mathrm{coef}(a_2)y_1^{m_2}y_2^{n_2} = \mathrm{coef}(c_0) + \mathrm{coef}(c_2)y_1^{m_2}y_2^{n_2} = 0. \tag{6.4.15}$$

Such a system has at most one positive solution. Indeed, since this is the case where the support is a simplex.

iii) Third case: $\alpha = \gamma_0 \leq \beta < \gamma_2$ (the case where $\alpha = \gamma_2 \leq \beta < \gamma_0$ is similar).

a) Assume first that $\alpha = \gamma_0 < \min(\beta, \gamma_2)$, then the reduced system of (6.4.14) with respect to v_0 becomes

$$\mathrm{coef}(a_0) + y_1^{m_1} + \mathrm{coef}(a_2)y_1^{m_2}y_2^{n_2} = \mathrm{coef}(c_0) - \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} = 0. \tag{6.4.16}$$

Such a system has at most two positive solutions. Indeed, since this can be reduced to an equation in one variable with at most three monomials.

b) Assume now that $\alpha = \gamma_0 = \beta < \gamma_2$. Then the reduced system of (6.4.14) with respect to v_0 becomes

$$\operatorname{coef}(a_0) + y_1^{m_1} + \operatorname{coef}(a_2) y_1^{m_2} y_2^{n_2} = \operatorname{coef}(c_0) - \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} = 0. \tag{6.4.17}$$

Such a system has at most five positive solutions. Indeed, since this is a system of two trinomials in two variables (see [LRW03]).

iv) Fourth case: $\alpha = \gamma_0 = \gamma_2 \leq \beta$.

a) Assume first that $\alpha = \gamma_0 = \gamma_2 < \beta$. Then the reduced system of (6.4.14) with respect to v_0 becomes

$$\begin{array}{lllllll} \operatorname{coef}(a_0) & + & \operatorname{coef}(a_2) y_1^{m_2} y_2^{n_2} & + & y_1^{m_1} & = 0, \\ \operatorname{coef}(c_0) & + & \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} & - & \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} & = 0. \end{array} \tag{6.4.18}$$

Such a system has at most three positive solutions. Indeed, since this is the case where the support is a circuit.

b) Assume now that $\alpha = \beta = \gamma_0 = \gamma_2$, then the reduced system of (6.4.14) with respect to v_0 becomes

$$\begin{array}{lllllll} \operatorname{coef}(a_0) & + \operatorname{coef}(a_2) y_1^{m_2} y_2^{n_2} & + & y_1^{m_1} & & = & 0, \\ \operatorname{coef}(c_0) & + \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} & - & \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} & + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} & = & 0. \end{array} \tag{6.4.19}$$

Such a system has at most eight real positive solutions if $\operatorname{coef}(a_0)/\operatorname{coef}(a_2) \neq \operatorname{coef}(c_0)/\operatorname{coef}(c_2)$ (see Proposition 6.53).

If $\operatorname{coef}(a_0)/\operatorname{coef}(a_2) = \operatorname{coef}(c_0)/\operatorname{coef}(c_2)$, then (6.4.19) has at most five positive solutions (again, see Proposition 6.53).

v) Fifth case: $\alpha = \beta < \min(\gamma_0, \gamma_2)$. The reduced system of (6.4.14) with respect to v_0 becomes

$$\operatorname{coef}(a_0) + y_1^{m_1} + \operatorname{coef}(a_2) y_1^{m_2} y_2^{n_2} = -\operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} = 0 \tag{6.4.20}$$

which has at most two real positive solutions (same argument as in the case **iii) b)**).

6.4.4 Type-(III) intersections outside the origin

Let v_0 denote the origin of $\mathcal{E}$. Lemma 6.26 shows that if T_1 and T_2 intersect non-transversally at a point v of type (III) such that $v \neq v_0$, then v belongs to the relative interior of a 1-cone of $\mathcal{E}$. In this Subsection, we write explicitly the reduced system of (6.4.1) with respect to v when $v \in \mathsf{L}_0$ or $v \in \mathsf{L}_1$. We explain in Section 6.6 why we omit the study of the reduced system of (6.4.1) with respect to v if it belongs to L_2. Moreover, we state Lemmata that give constraints on the tropical curves intersecting at a type-(III) point.

• Assume that $v \in \mathsf{L}_1$. Then the reduced system with respect to v becomes

$$y_1^{m_1} + \mathrm{coef}(a_2)y_1^{m_2}y_2^{n_2} + \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} = y_1^{m_1} + \mathrm{coef}(b_2)y_1^{m_2}y_2^{n_2} + \mathrm{coef}(b_4)y_1^{m_4}y_2^{n_4} = 0. \quad (6.4.21)$$

Note that if $\mathrm{coef}(a_2) = \mathrm{coef}(b_2)$ and (6.4.21) has a positive solution $(\alpha, \beta) \in (\mathbb{R}^*)^2$, then α is a solution of

$$y_1^{m_1} + d_2 y_1^{\frac{m_2(n_3-n_4)+n_2(m_4-m_3)}{n_3-n_4}} + d_3 y_1^{\frac{n_3m_4-m_3n_4}{n_3-n_4}} = 0, \quad (6.4.22)$$

and (α, β) satisfy

$$\beta = \left(\frac{\mathrm{coef}(b_4)}{\mathrm{coef}(a_3)}\right)^{1/(n_3-n_4)} \alpha^{\frac{m_4-m_3}{n_3-n_4}}, \quad (6.4.23)$$

with

$$d_2 = \mathrm{coef}(a_2)\left(\frac{\mathrm{coef}(b_4)}{\mathrm{coef}(a_3)}\right)^{n_2/(n_3-n_4)} \quad \text{and} \quad d_3 = \mathrm{coef}(a_3)\left(\frac{\mathrm{coef}(b_4)}{\mathrm{coef}(a_3)}\right)^{n_3/(n_3-n_4)}.$$

• Assume now that v belongs to L_0. Then the reduced system with respect to v becomes

$$\mathrm{coef}(a_0) + y_1^{m_1} + \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} = \mathrm{coef}(b_0) + y_1^{m_1} + \mathrm{coef}(b_4)y_1^{m_4}y_2^{n_4} = 0. \quad (6.4.24)$$

Similarly, if $\mathrm{coef}(a_0) = \mathrm{coef}(b_0)$ and (6.4.24) has a positive solution $(\alpha, \beta) \in (\mathbb{R}^*)^2$, then α is a solution of

$$\mathrm{coef}(a_0) + y_1^{m_1} + d_3 y_1^{\frac{n_3m_4-m_3n_4}{n_3-n_4}} = 0. \quad (6.4.25)$$

and (α, β) satisfy (6.4.23).

Remark 6.42. *Both* (6.4.21) *and* (6.4.24) *have four monomials in their support, thus each of them has at most three positive solutions. On the other hand, following Descartes' rule of signs, each of* (6.4.22) *and* (6.4.25) *has at most two positive solutions.*

The following Lemmata will be useful in the next Sections. Recall that we assumed that (6.4.1) is highly non-degenerate.

Lemma 6.43. *The tropical curves T_1 and T_2 have at most one intersection point of type (III), different from the origin.*

Proof. Assume without loss of generality that T_1 and T_2 intersect at two points v_1 and v_2 of type (III) such that $v_1 \in \mathsf{L}_1$ and $v_2 \in \mathsf{L}_2$. Lemma 6.20 shows that, since both v_1 and v_2 are vertices of T_1 and T_2, the tropical curve T_1 (resp. T_2) has an edge $F_{2,3}$ (resp. $F_{2,4}$) adjacent to both v_1 and v_2. Therefore, we have $F_{2,3} = F_{2,4}$, and thus it is a non-transversal intersection of type (I) in C_2. This implies that the segments $[w_2, w_3] \in \tau_1$ and $[w_2, w_4] \in \tau_2$ are parallel, which contradicts that (6.4.1) is highly non-degenerate. (see Figure 6.8). □

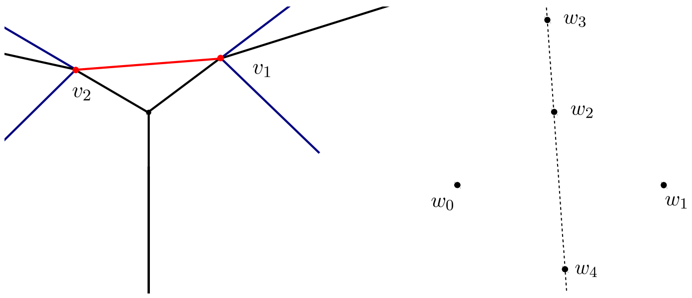

Figure 6.8: An example showing that if T_1 and T_2 intersect non-transversally at two points of type (III), then the system (6.4.1) is not highly non-degenerate.

Lemma 6.44. *Assume that T_1 and T_2 intersect non-transversally at a point $v \neq v_0$ of type (III). Then T_1 and T_2 intersect transversally in at most one point. Moreover, if this is the case, then this transversal intersection point is not contained in a 2-cone of $\mathcal{E}$ adjacent to v (see Figure 6.9).*

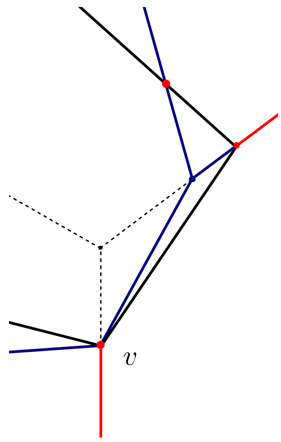

Figure 6.9: The tropical curves T_1 and T_2 intersect transversally at only one point belonging to C_2.

Proof. Assume that T_1 and T_2 intersect at a point $v \in \mathring{\mathsf{L}}_0$ of type (III). Since v is a common vertex of T_1 and T_2, applying Corollary 6.19 and Lemma 6.22 to T_1 and T_2, we get that C_0 and C_1 do not contain transversal intersection points of T_1 and T_2. Moreover, Theorem 6.15 shows that C_2 contains at most one transversal intersection. □

6.5 Proof of Theorem 6.1

In all what follows, we assume that $(\alpha, \beta) \neq (0, 0)$, and consider the highly non-degenerate normalized system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0. \end{aligned} \tag{6.5.1}$$

satisfying that all a_i and b_j are in $\mathbb{RK}^*$ and verify $\mathrm{ord}(a_i) = \mathrm{ord}(b_j) = 0$, all w_i are in $\mathbb{Z}^2$, both m_1, n_2 are positive and both α, β are real numbers.

Recall that since $\mathcal{E}$ is a base fan of (6.5.1), then the possible intersection components of the tropical curves T_1 and T_2, associated to the first and second equations respectively, are the following.

1. The set of transversal intersection points, denote it by $\mathfrak{T}$.
2. A set of at most three non-transversal intersections of type (I), satisfying that for $i \in \{0, 1, 2\}$, a 1-cone $\mathring{\mathsf{L}}_i$ of $\mathcal{E}$ contains at most one type-(I) intersection, denoted it by $\mathfrak{E}_i$.
3. The set of non-transversal intersection points of type (II) , denote it by $\mathfrak{N}_2$.
4. The origin of the base fan $\mathcal{E}$, denote it by v_0.
5. A non-transversal intersection point of type (III), outside the origin of $\mathcal{E}$, denote it by v. There can be at most one of such type since (6.5.1) is highly non-degenerate.

We have the following two results.

Lemma 6.45. *The (possibly empty) set* $\{v\} \cup \mathfrak{T}$ *contains the valuations of at most four non-degenerate positive solutions of* (6.5.1).

Proof. If T_1 and T_2 do not intersect non-transversally at a point v of type (III) outside the origin of $\mathcal{E}$, then Theorem 6.15 shows that (6.5.1) has at most three non-degenerate positive solutions with valuation in $\mathfrak{T}$. Otherwise, the result comes from Remark 6.42 and Lemma 6.44. □

Proposition 6.46. *If* $\alpha \neq \beta$ *or* $\alpha = \beta < 0$*, then the set* $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \mathfrak{N}_2 \cup \{v_0\}$ *contains the valuations of at most five positive solutions of* (6.5.1).

Proof. Assume that $\alpha \neq \beta$ or $\alpha = \beta < 0$.

• Assume first that $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ for $i = 0, 2$. Then, a consequence of Corollary 6.40 gives that any intersection point of type (II) is not a valuation of a non-degenerate positive solution of (6.5.1). Moreover, since we do *not* have $\alpha = \beta > 0$, then the origin v_0 of $\mathcal{E}$ is the valuation of at most three non-degenerate positive solutions. Indeed, this comes from the analysis done in Subsection 6.4.3, where the possible case that gives the biggest sharp bound is **iv) a)** with $0 < \alpha = \gamma_0 = \gamma_2 < \beta$. If (6.5.1) does not have non-degenerate positive solutions with valuations in the relative interior of an intersection cell of type (I), then $\{v_0\}$ is the only element of the set $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \mathfrak{N}_2 \cup \{v_0\}$ that contains the valuations of non-degenerate positive solutions of (6.5.1), and we are done.

Assume that (6.5.1) has non-degenerate positive solutions with valuations contained in the relative interiors of intersection cells of type (I). Then Lemma 6.33 shows that the relative interior of at least one intersection cell of type (I), say $\mathfrak{E}_1 \subset \mathsf{L}_1$, does not contain valuations of non-degenerate positive solutions of (6.5.1). Similarly as in Subsection 6.4.3, we study here four cases with respect to the values of α, β, γ_0 and γ_2. Recall that $f_{0,t}$ and $f_{2,t}$ in (6.4.9) and (6.4.10) respectively are approximation polynomials of (6.5.1) for $\mathfrak{E}_0$ and $\mathfrak{E}_2$ respectively, and that $f_{0,t}$ and $f_{2,t}$ have at most three and two non-degenerate positive roots respectively. We keep the notations for Γ_0 and Γ_2 as the lower hulls of the Newton polytopes of the Viro approximation polynomials $f_{0,t}$ and $f_{2,t}$ respectively. We apply Corollary 6.12 by counting in each case the number of edges of Γ_0 and Γ_2 with negative slope. We will deduce after each of the following cases that the set $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\}$ contains the valuations of at most five non-degenerate positive solutions of (6.6.2), and thus the same goes for $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \mathfrak{N}_2 \cup \{v_0\}$.

i) First case: there exists only one element of the set $\{\alpha, \beta, \gamma_0, \gamma_2\}$ that is equal to $\min(\alpha, \beta, \gamma_0, \gamma_2)$. Then (6.5.1) does not have non-degenerate positive solutions with valuation v_0 (since in any case, the second equation of (6.4.14) has only one monomial). Therefore, the lower hulls Γ_0 and Γ_2 has at most three (resp. two) edges with negative slope, and thus the set $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$ contains the valuations of at most five non-degenerate positive solutions of (6.5.1).

ii) Second case: $\gamma_0 = \gamma_2 < \min(\alpha, \beta)$. Then (6.5.1) has at most one non-degenerate positive solution with valuation v_0. Moreover, the relative interior $\mathring{\mathfrak{E}}_0$ of $\mathfrak{E}_0$ has at most two non-degenerate positive solutions since the lower hull Γ_0, associated to f_0, has at most two edges with negative slope. Therefore, the system (6.5.1) has at most four non-degenerate positive solutions with valuation in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$.

iii) a) Third case: $\alpha = \gamma_0 < \min(\beta, \gamma_2)$ (the case where $\alpha = \gamma_2 < \min(\beta, \gamma_0)$ is similar). Then (6.5.1) has at most two non-degenerate positive solution with valuations v_0 (case of a trinomial and a binomial). Moreover, $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$) has at most two (resp. one) non-degenerate positive solutions since the lower hull Γ_0 (resp. Γ_2), associated to $f_{0,t}$ (resp. $f_{2,t}$), has at most two (resp. one) edges with negative slope. Therefore, the system (6.5.1) has at most three non-degenerate positive solutions with valuation in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$.

iv) a) Fourth case: $\alpha = \gamma_0 = \gamma_2 < \beta$. Then (6.5.1) has at most three non-degenerate positive solution with valuation v_0. Moreover, for $i = 0, 2$, $\mathring{\mathfrak{E}}_i$ has at most one non-degenerate positive solution of (6.4.1) since the lower hull Γ_i, associated to $f_{i,t}$ has at most one edge with negative slope. Therefore, the system (6.5.1) has at most two non-degenerate positive solutions with valuation in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$.

This finishes the proof for the case where $\operatorname{coef}(a_i) = \operatorname{coef}(b_i)$ for $i = 0, 2$.

• Assume now that $\operatorname{coef}(a_0)/\operatorname{coef}(b_0) \neq \operatorname{coef}(a_2)/\operatorname{coef}(b_2)$ and $\operatorname{coef}(a_i) \neq \operatorname{coef}(b_i)$ for $i = 0, 2$ (see Remark 6.32). Note that from the beginning of this section, we have $\alpha\beta \neq 0$. Then v_0 is the valuation of at most one non-degenerate positive solution of (6.5.1) (since the reduced system is supported on a simplex). Moreover, the system (6.5.1) does not have any solutions with valuation in any $\mathring{\mathfrak{E}}_i$ for $i \in \{0, 1, 2\}$. Indeed, since from $\operatorname{coef}(a_0) \neq \operatorname{coef}(b_0)$, the reduced system with respect to $\mathring{\mathfrak{E}}_0$ for example, is

$$\operatorname{coef}(a_0) + y_1^{m_1} = \operatorname{coef}(b_0) + y_1^{m_1} = 0$$

and thus has no solutions.

The tropical curves T_1 and T_2 intersect in at most five non-transversal intersection points of type (II). Indeed, since T_1 (resp. T_2) has at most three vertices outside v_0, and this happens only when α (resp. β) is negative. Moreover, if α and β are both negative or positive, then T_1 and T_2 intersect in at most three points of type (II) (see Figure 6.10 for example).

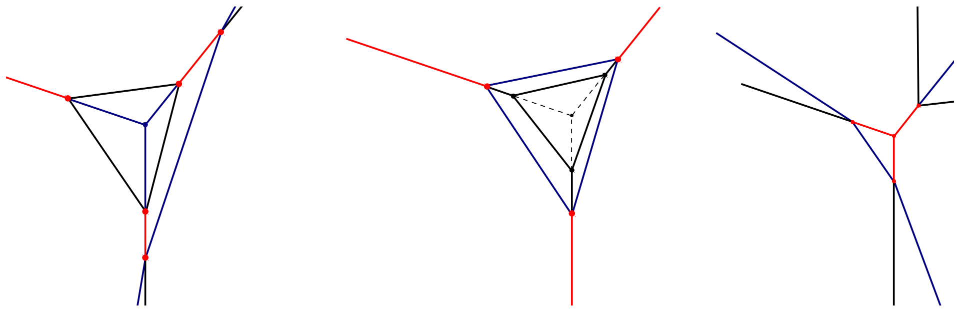

Figure 6.10: Possible restrictions for T_1 and T_2 with respect to α, β. From left to right: $\alpha < 0 < \beta$, $\alpha, \beta < 0$ and $\alpha, \beta > 0$.

Therefore, if T_1 and T_2 intersect at five points of type (II), then these two curves do not intersect at the origin v_0 of $\mathcal{E}$, since one would require that $\alpha, \beta > 0$. This finishes the proof. $\square$

The following corollary proves Theorem 6.1 for the case where $\alpha \neq \beta$ or $\alpha = \beta < 0$.

Corollary 6.47. *If $\alpha \neq \beta$ or $\alpha = \beta < 0$, then the set*

$$\mathfrak{T} \cup \mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \mathfrak{N}_2 \cup \{v_0\} \cup \{v\}$$

contains the valuations of at most nine positive solutions of (6.5.1).

Proof. If $(\alpha, \beta) \neq (0, 0)$, by Lemma 6.45, the set $\mathfrak{T} \cup \{v\}$ contains the valuations of at most four positive solutions of (6.4.1). By Proposition 6.46, if in addition we have $\alpha \neq \beta$ or $\alpha = \beta < 0$, then the set $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \mathfrak{N}_2 \cup \{v_0\}$ contains the valuations of at most five positive non-degenerate solutions of (6.4.1). $\square$

In what follows, we assume that $\alpha = \beta > 0$. If $\operatorname{coef}(a_i) \neq \operatorname{coef}(b_i)$ for $i = 0, 2$ and $\operatorname{coef}(a_0)/\operatorname{coef}(b_0) \neq \operatorname{coef}(a_2)/\operatorname{coef}(b_2)$, and $\alpha\beta \neq 0$ (see Remark 6.32), and thus Theorem 6.1 comes easy. Indeed, Lemma 6.45 and the second part of the proof of Proposition 6.46 also apply to this case, and thus so does Corollary 6.47.

We assume furthermore in what follows that $\operatorname{coef}(a_i) = \operatorname{coef}(b_i)$ for $i = 0, 2$, thus the normalized system (6.4.1) becomes

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\alpha} y_1^{m_4} y_2^{n_4} &= 0. \end{aligned} \tag{6.5.2}$$

In this section, we prove the following result.

Theorem 6.48. *The system* (6.5.2) *has at most nine non-degenerate positive solutions. Moreover, there exists a system* (6.5.2) *that has seven non-degenerate positive solutions.*

We first show why the first statement of Theorem 6.48 is trivial if both $\operatorname{coef}(a_0)$ and $\operatorname{coef}(a_2)$ are positive. Note that the reduced system of (6.5.2) with respect to the origin will not have positive solutions. Indeed, since the reduced system of (6.5.2) with respect to the origin will have the equation $\operatorname{coef}(a_0) + y_1^{m_1} + \operatorname{coef}(a_2) y_1^{m_2} y_2^{n_2}$, which has no positive solutions. If T_1 and T_2 intersect non-transversally at a cell of type (I), the relative interior of such a cell does not contain the valuations of positive solutions of (6.5.2) (this follows from $\operatorname{coef}(a_0), \operatorname{coef}(a_2) > 0$ as in the proof of Lemma 6.33 for example). Moreover, we deduce from Corollary 6.40 that (6.5.2) does not have non-degenerate positive solutions with valuations non-transversal intersection points of type (II). Therefore, the only cells of T_1 and T_2 that can contain the valuations of non-degenerate positive solutions of (6.5.2) are transversal intersection points and non-transversal intersection points of type (III) that are different from $(0,0)$. Theorem 6.15 shows that (6.5.2) has at most three positive solutions with valuations transversal intersection points of T_1 and T_2. Therefore, if there does not exist a non-transversal intersection point of type (III) in the relative interior of a 1-cone of $\mathcal{E}$, then (6.5.2) has at most three positive solutions. Otherwise, if there exists a non-transversal intersection point $v \neq (0,0)$ of type (III), then Remark 6.42 and Lemma 6.44 show that (6.5.2) has at most three positive solutions, and we are done.

In what follows, we assume that $\operatorname{coef}(a_0) < 0$ and $\operatorname{coef}(a_2) > 0$ are not both positive. Note that if $\operatorname{coef}(a_0), \operatorname{coef}(a_2) < 0$, or $\operatorname{coef}(a_0) > 0$ and $\operatorname{coef}(a_2) < 0$, one can associate to (6.5.2) a normalized system similar to (6.5.2) that has the same number of non-degenerate positive solutions as (6.5.2) and satisfying $\operatorname{coef}(a_0) < 0$ and $\operatorname{coef}(a_2) > 0$. This is done via monomial change of coordinates and multiplying the equations of (6.5.2) by some terms (as the ones made in the proof of Lemma 6.31 for example).

Multiplying each polynomial of (6.5.2) by some real number and making some change of coordinates if necessary (see the proof of Proposition 6.6 for example), we may assume that

$$\operatorname{coef}(a_0) = -1 \quad \text{and} \quad \operatorname{coef}(a_2) = 1. \tag{6.5.3}$$

6.5.1 First part of Theorem 6.48

In this subsection, we prove the following result.

Proposition 6.49. *The system* (6.5.2) *cannot have more than nine positive solutions.*

Let Δ_1 and Δ_2 (resp. τ_1 and τ_2, T_1 and T_2) denote the Newton polytopes (resp. dual subdivisions, tropical curves) associated to the first and second equation of (6.5.2) respectively.

Lemma 6.50. *The curves T_1 and T_2 cannot intersect transversally at more than one point.*

Proof. Assume that T_1 and T_2 intersect transversally at two points p_0 and p_1, we prove that this gives a contradiction. We treat the case $p_0 \in \mathsf{C}_0$ and $p_1 \in \mathsf{C}_1$ (the other cases are symmetric). Using Lemma 6.28, we compute the coordinates of p_0 and p_1 to obtain $k_0(n_4 - n_3\ ,\ m_3 - m_4)$ and $k_1(n_4 - n_3\ ,\ m_3 - m_4)$ respectively, with

$$k_0 = \frac{\alpha}{m_3 n_4 - m_4 n_3} \quad \text{and} \quad k_1 = \frac{\alpha}{m_3 n_4 - m_4 n_3 - m_1(n_4 - n_3)}.$$

Note that since $p_0 \in \mathsf{C}_0$ and $p_1 \in \mathsf{C}_1$, we have $k_0(n_4 - n_3) < 0$ and $k_1(n_4 - n_3) > 0$. Indeed, the 1-cone L_0 (which is adjacent to both C_0 and C_1) belongs to a vertical line passing through the origin $(0,0)$ of $\mathcal{E}$.

Assume that $m_3n_4 - m_4n_3 > 0$, then since $k_0k_1 < 0$ (from $k_0(n_4 - n_3) < 0$ and $k_1(n_4 - n_3) > 0$), we obtain $m_3n_4 - m_4n_3 - m_1(n_4 - n_3) < 0$ from the expressions of k_0 and k_1. Therefore, from $0 < m_3n_4 - m_4n_3 < m_1(n_4 - n_3)$ and $m_1 > 0$, we obtain $0 < n_4 - n_3$. We deduce from $k_0(n_4 - n_3) < 0$ that k_0 is negative, which makes α also negative, a contradiction. Similarly, we arrive at a contradiction when assuming that $m_3n_4 - m_4n_3 < 0$. □

Lemma 6.51. *If T_1 and T_2 intersect non-transversally at a point $v \neq (0,0)$ of type (III), then T_1 and T_2 do not intersect transversally at a point, and the reduced system with respect to v has at most one positive solution.*

Proof. Assume that T_1 and T_2 intersect non-transversally at a point $v \neq (0,0)$ of type (III) and transversally at a point p, we prove that this gives a contradiction. Assume without loss of generality that $v \in \mathsf{L}_0$. Since T_1 and T_2 have vertices in L_0 that coincide, from the equality $\alpha/n_3 = \alpha/n_4$ (see the beginning of Section 6.4), we deduce that $n_3 = n_4$. Moreover, since $\alpha > 0$ and $v \in \mathsf{L}_0$, we have $n_3 = n_4 < 0$. On the other hand, Lemma 6.44 shows that $p \in \mathsf{C}_2$, thus by Lemma 6.28, the coordinates (x_1, x_2) of p verify

$$m_2x_1 + n_2x_2 = m_3x_1 + n_3x_2 - \alpha = m_4x_1 + n_3x_2 - \alpha.$$

A simple computation shows that $p = (0, \alpha/(n_3 - n_2))$, and thus $\alpha/(n_3 - n_2) > 0$. Indeed, since otherwise we get that the transversal intersection point p belongs to L_0, contradicting Theorem 6.15. Recall that $n_2 > 0$ ((6.5.2) is a normalized system). Now, since $\alpha > 0$ and $\alpha/(n_3 - n_2) > 0$, we get $n_3 - n_2 > 0$, a contradiction to $n_3 < 0 < n_2$.

As for the second part of the Lemma, the reduced system with respect to v is

$$-1 + y_1^{m_1} + \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} = -1 + y_1^{m_1} + \mathrm{coef}(b_4)y_1^{m_4}y_2^{n_3} = 0, \tag{6.5.4}$$

and has at most one positive solution. Indeed, assume that (ρ_1, ρ_2) is a positive solution of the latter system. Taking the difference of two equations we get $\mathrm{coef}(a_3)\rho_1^{m_3} = \mathrm{coef}(b_4)\rho_1^{m_4}$, and thus $\rho_1 = (\mathrm{coef}(a_3)/\mathrm{coef}(b_4))^{1/(m_4 - m_3)}$. Plugging it in the first equation of (6.5.4), we retrieve only one value for ρ_2. □

Note that since $\alpha > 0$, the curves T_1 and T_2 intersect non-transversally at the apex of $\mathcal{E}$ (see Figure 6.11 for example). Furthermore, these curves intersect at three cells $\mathfrak{E}_0$, $\mathfrak{E}_1$ and $\mathfrak{E}_2$ of type (I) contained in L_0, L_1 and L_2 respectively. Denote again the apex of $\mathcal{E}$ by v_0. It follows from Corollary 6.40 that since $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ for $i = 0, 2$, the system (6.5.2) does not have a positive solution with valuation at a point of type (II). Since $\mathrm{coef}(a_2) > 0$, the reduced system $y_1 + \mathrm{coef}(a_2)y_1^{m_2}y_2^{n_2} = 0$ does not have positive solutions (see Proof of Lemma 6.33 for example), thus $\mathring{\mathfrak{E}}_1$ does not contain valuations of positive solutions of (6.5.2). Lemmata 6.50 and 6.51 show that whether T_1 and T_2 intersect non-transversally at point $v \neq v_0$ of type (III) or not, the set $\mathfrak{D} := T_1 \cap T_2 \setminus (\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\})$ contains the valuations of at most one positive solution of (6.5.2).

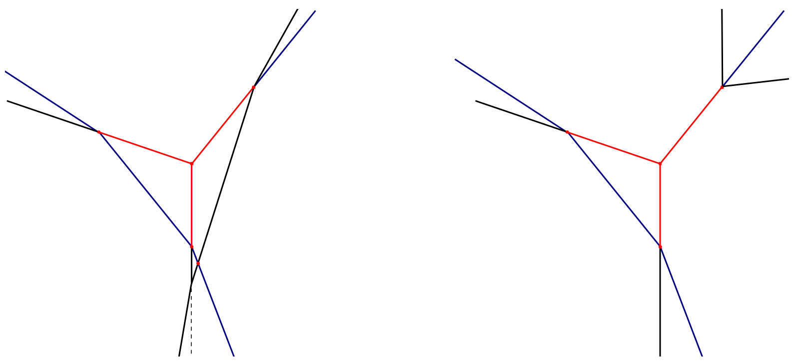

Figure 6.11: Examples showing that the curves T_1 and T_2 intersect at the apex v_0 of $\mathcal{E}$ and at three cells of type (I).

From Subsection 6.4.3, the number of positive solutions of (6.5.2) with valuation v_0 is equal to the number of positive solutions of the reduced system of

$$\begin{aligned} -1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ c_0 t^{\gamma_0} + c_2 t^{\gamma_2} y_1^{m_2} y_2^{n_2} - a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} + b_4 t^{\alpha} y_1^{m_4} y_2^{n_4} &= 0 \end{aligned} \tag{6.5.5}$$

with respect to v_0, with $c_i t^{\gamma_i} = b_i - a_i$, $\text{ord}(c_i) = 0$ and $\gamma_i \geq 0$ for $i = 0, 2$. Recall from Subsection 6.4.1 that

$$f_{0,t} = \text{coef}(c_0) t^{\gamma_0} + \text{coef}(c_2) t^{\gamma_2} y^{n_2} - \text{coef}(a_3) t^{\alpha} y^{n_3} + \text{coef}(b_4) t^{\alpha} y^{n_4} \tag{6.5.6}$$

and

$$f_{2,t} = ct^{\delta} - \text{coef}(a_3) t^{\alpha} y^{\frac{m_3 n_2 - m_2 n_3}{n_2}} + \text{coef}(b_4) t^{\alpha} y^{\frac{m_4 n_2 - m_2 n_4}{n_2}}, \tag{6.5.7}$$

with $c_i = b_i - a_i$, $\gamma_i = \text{ord}(c_i)$ for $i = 0, 2$ and ct^{δ} is the first-order term of $c_2 - c_0$, are approximation polynomials of (6.5.2) for $\mathfrak{E}_0$ and $\mathfrak{E}_2$ respectively. We deduce from Corollary 6.12 that the number of non-degenerate positive solutions of (6.5.2) with valuation in $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$) is less or equal to the number of non-degenerate roots $\mathbb{R}\mathbb{K}^*_{>0}$ of $f_{0,t}$ (resp. $f_{2,t}$) with positive order and that are also largely ordered (see Definition 6.9) with respect to $f_{0,t}$ (resp. $f_{2,t}$). The first order terms of all such roots of $f_{0,t}$ and $f_{2,t}$ are completely determined from some edges of Γ_0 and Γ_2 with negative slope together with their respective facial subpolynomials.

Remark 6.52. *In what follows, by "*edge*" of the lower hull Γ_0 (resp. Γ_2), we mean a segment of Γ_0 (resp. Γ_2) that supports only a binomial.*

In 6.5.1.1, we make an analysis on $f_{0,t}$, $f_{2,t}$ and (6.5.5) with respect to the different possibilities of equalities and inequalities between α, γ_0 and γ_2. The results obtained in 6.5.1.1 can be summarized in the following two tables. The numbers appearing in the entries of these tables represent the maximum number of positive solutions of (6.5.2) with valuations in the associated intersection components of $T_1 \cap T_2$. In fact, the non-zero entries in the row $\mathfrak{E}_0$ (resp. $\mathfrak{E}_2$) correspond to the maximal numbers of edges of Γ_0 (resp. Γ_2) with negative slope.

Intersection Locus	$\gamma_0 \neq \gamma_2$ and $\min(\gamma_0, \gamma_2) < \alpha$	$\gamma_0 = \gamma_2 < \alpha$	$\alpha < \min(\gamma_0, \gamma_2)$
$\mathfrak{D}$	1	1	1
$\mathfrak{E}_0$	2	1	2
$\mathfrak{E}_2$	1	1	1
$\{v_0\}$	0	1	2

Table 6.1: $\alpha \neq \gamma_i$ for $i = 0, 2$.

Intersection Locus	$\alpha = \gamma_0 < \gamma_2$	$\alpha = \gamma_2 < \gamma_0$	$\alpha = \gamma_0 = \gamma_2$ $\mathrm{coef}(c_0) = \mathrm{coef}(c_2)$	$\alpha = \gamma_0 = \gamma_2$ $\mathrm{coef}(c_0) \neq \mathrm{coef}(c_2)$
$\mathfrak{D}$	1	1	1	1
$\mathfrak{C}_0$	0	1	0	0
$\mathfrak{C}_2$	0	0	1	0
$\{v_0\}$	5	5	5	8

Table 6.2: $\alpha = \gamma_i$ for $i \in \{0, 2\}$.

The bound $8+1 = 9$ for the number of non-degenerate positive solutions of (6.5.2) is the largest among all other possible cases shown in the latter tables. This finishes the proof of Proposition 6.49 given that the entries of the tables are correct.

6.5.1.1 Proof that the entries of the tables (6.1) and (6.2) are correct

We make an analysis similar to that formulated in Subsection 6.4.3 on $f_{0,t}$, $f_{2,t}$ and on all possible reduced systems of (6.5.5) with respect to v_0. Assume without loss of generality that $n_3 < n_4$.

First, we note that Γ_0 is the lower part of the convex hull of points in

$$\{(0, \gamma_0),\ (n_2, \gamma_2),\ (n_3, \alpha),\ (n_4, \alpha)\}.$$

Since (n_3, α) and (n_4, α) have the same second coordinate, clearly Γ_0 has at most two edges with negative slope. The same goes for Γ_2, which is the lower part of the convex hull of at most three points among

$$\{(0, \delta),\ ((m_3n_2 - m_2n_3)/n_2, \alpha)\,,\ ((m_4n_2 - m_2n_4/n_2, \alpha)\}.$$

Thus Γ_2 has at most one edge with negative slope.

i) First case: $\gamma_0 \neq \gamma_2$ and $\min(\gamma_0, \gamma_2) < \alpha$. Then the reduced system of (6.5.5) with respect to v_0 has no positive solutions (since in any case, the second equation has only one monomial). We see an example in Figure 6.12 of Γ_0 and Γ_2.

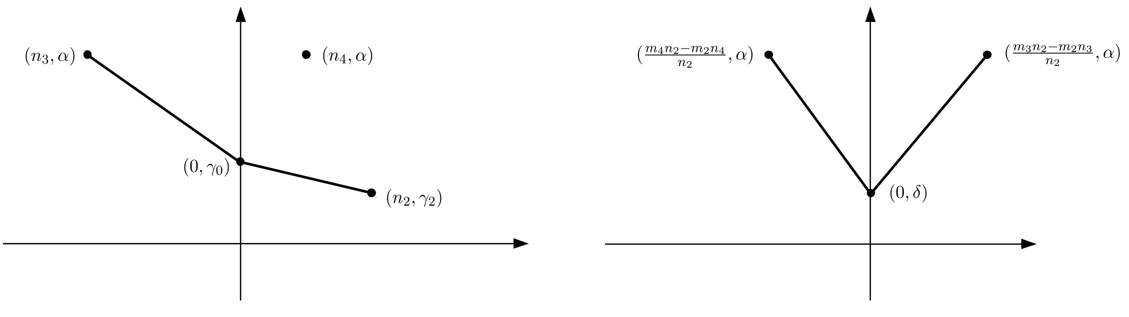

Figure 6.12: The graphs Γ_0 and Γ_2 in the first case.

ii) Second case: $\gamma_0 = \gamma_2 < \alpha$. Then the reduced system of (6.5.5) with respect to v_0 is

$$-1 + y_1^{w_1} + y_1^{m_2} y_2^{n_2} = \mathrm{coef}(c_0) + \mathrm{coef}(c_2) y_1^{m_2} y_2^{n_2} = 0,$$

which has at most one positive solution (this is deduced by replacing $y_1^{m_2} y_2^{n_2}$ by $-\operatorname{coef}(c_2)/\operatorname{coef}(c_0)$ in the first equation of the latter system). Since the points $(0, \gamma_0)$ and (n_2, γ_2) have the same second coordinate, the lower hull Γ_0 has at most one edge with negative slope (see Figure 6.13 on the left for example).

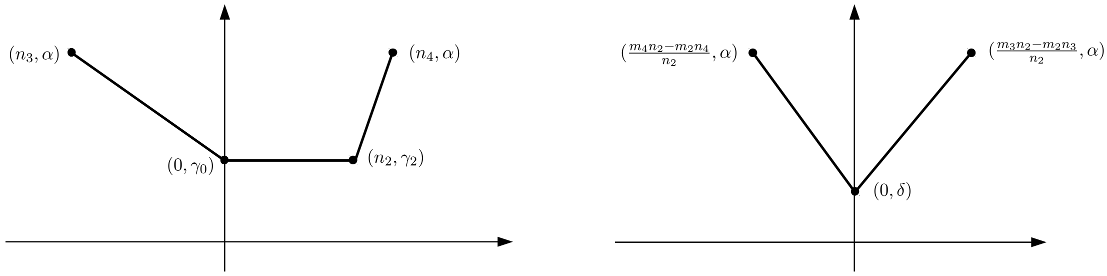

Figure 6.13: The graphs Γ_0 and Γ_2 in the second case.

iii) Third case: $\gamma_2 = \alpha < \gamma_0$ (The case where $\gamma_0 = \alpha < \gamma_2$ is similar). The reduced system of (6.5.5) with respect to v_0 is

$$-1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} = \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} - \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} = 0,$$

which has at most five positive solutions (since this system is of two trinomials in two variables). The lower hull Γ_0 has at most one edge with negative slope (see Figure 6.14 on the left). Recall that $\delta = \operatorname{ord}(c_2 - c_0)$. Thus, since $\gamma_2 = \alpha < \gamma_0$, we get $\delta = \gamma_2 < \gamma_0$ which implies that Γ_2 is a horizontal edge (see Figure 6.14 on the right).

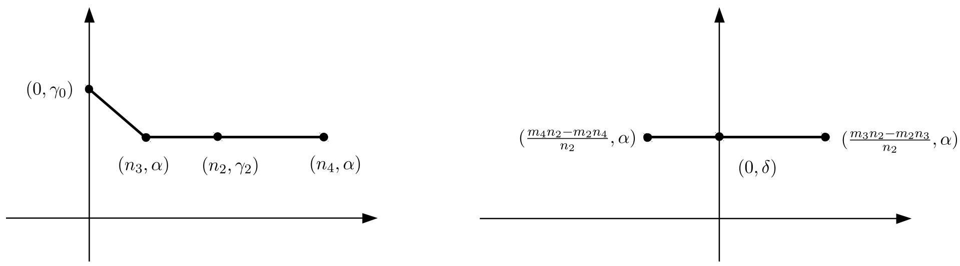

Figure 6.14: The graphs Γ_0 and Γ_2 in the third case.

iv) Fourth case: $\gamma_2 = \alpha = \gamma_0$. The lower hull Γ_0 is a horizontal segment (see Figure 6.15 on the left). Then the reduced system of (6.5.5) with respect to v_0 is

$$\begin{array}{ccccccccc} -1 & + y_1^{m_2} y_2^{n_2} & + & y_1^{m_1} & & & = & 0, \\ \operatorname{coef}(c_0) & + \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} & - & \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} & + & \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} & = & 0. \end{array} \qquad (6.5.8)$$

We distinguish two cases:

1. Assume that $\operatorname{coef}(c_0) = \operatorname{coef}(c_2)$. Then (6.5.8) has at most five positive solutions (see Proposition 6.53). Since the first-order term of $c_2 - c_0$ is ct^δ, from $\operatorname{coef}(c_0) = \operatorname{coef}(c_2)$, $\operatorname{ord}(c_0) = \operatorname{ord}(c_2) = \gamma_0 = \gamma_2$, we get $\delta > \gamma_0 = \gamma_2 = \alpha$. Therefore, the lower hull Γ_2 has at most one edge with negative slope (see Figure 6.15 on the right).

2. Assume that $\operatorname{coef}(c_0) \neq \operatorname{coef}(c_2)$. Then (6.5.8) has at most eight positive solutions (see Proposition 6.53). Since the first-order term of $c_2 - c_0$ is ct^δ, from $\operatorname{coef}(c_0) \neq \operatorname{coef}(c_2)$, we get $\delta = \gamma_0 = \gamma_2 = \alpha$. Therefore, the lower hull Γ_2 is a horizontal line (see Figure 6.14 on the right).

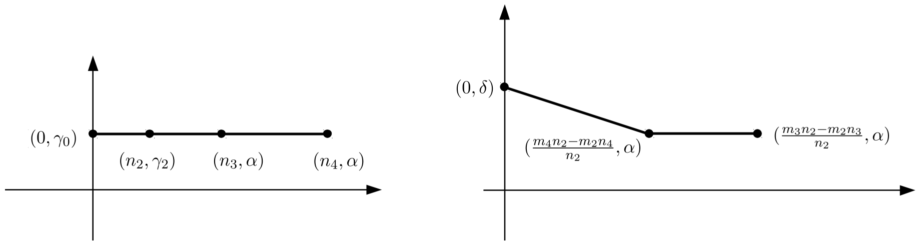

Figure 6.15: The graphs Γ_0 and Γ_2 in the fourth case.

v) Fifth case: $\alpha < \min(\gamma_0, \gamma_2)$. Then the reduced system of (6.5.5) with respect to v_0 becomes

$$-1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} = -\operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} = 0,$$

which has at most two positive solutions.

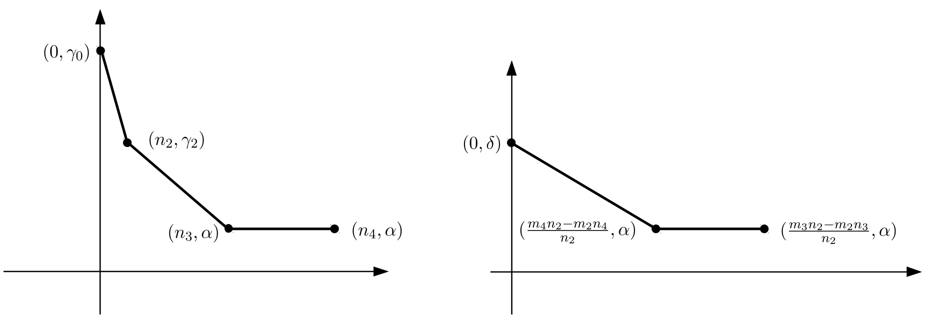

Figure 6.16: The graphs Γ_0 and Γ_2 in the fifth case.

Consider the real polynomial system

$$\begin{array}{lllllll} -1 & + y_1^{m_2} y_2^{n_2} & + & y_1^{m_1} & & = & 0, \\ \operatorname{coef}(c_0) & + \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} & - & \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} & + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} & = & 0, \end{array} \qquad (6.5.9)$$

with support in $\mathbb{Z}^2$, where both m_1 and n_2 are positive integers. □

Proposition 6.53. *If* $\mathrm{coef}(c_0) = \mathrm{coef}(c_2)$*, then* (6.5.9) *has at most five positive solutions. Moreover, if* $\mathrm{coef}(c_0) \neq \mathrm{coef}(c_2)$*, then* (6.5.9) *has at most eight positive solutions.*

Proof. For the first statement. Without loss of generality, suppose that $\mathrm{coef}(c_0) < 0$. Then, the system

$$\begin{array}{rcrcrcl} -1 & + & y_1^{m_2} y_2^{n_2} & + & y_1^{m_1} & = & 0, \\ -\dfrac{\mathrm{coef}(a_3)}{\mathrm{coef}(c_2)} y_1^{m_3} y_2^{n_3} & + & \dfrac{\mathrm{coef}(b_4)}{\mathrm{coef}(c_2)} y_1^{m_4} y_2^{n_4} & - & y_1^{m_1} & = & 0, \end{array} \tag{6.5.10}$$

has the same number of non-degenerate positive solutions as (6.5.9). Indeed, the second equation of (6.5.10) is obtained by dividing the second equation of (6.5.9) by $\mathrm{coef}(c_2)$, and subtracting from it the first equation of (6.5.9). The system (6.5.10) is a system of two trinomials in two variables, thus it has at most five positive non-degenerate solutions.

For the second statement. Assume now that $\mathrm{coef}(c_0) \neq \mathrm{coef}(c_2)$. We look for the positive solutions of (6.5.9). The first equation of this system may be written as $y_2 = x^{\alpha}(1-x)^{\beta}$, where $x = y_1^{m_1}$, $\alpha = -m_2/(m_1 n_2)$ and $\beta = 1/n_2$. It is clear that $y_1, y_2 > 0 \Leftrightarrow x \in I_0 =]0,1[$. Plugging y_1 and y_2 in the second equation of (6.5.9), we get the equation $f = 0$, with

$$f(x) = \mathrm{coef}(c_0) + \mathrm{coef}(c_2) - \mathrm{coef}(c_2)x - \mathrm{coef}(a_3)x^{\alpha_3}(1-x)^{\beta_3} + \mathrm{coef}(b_4)x^{\alpha_4}(1-x)^{\beta_4},$$

$\alpha_i := \dfrac{m_i n_2 - m_2 n_i}{m_1 n_2}$ and $\beta_i := \dfrac{n_i}{n_2}$ for $i = 3, 4$. The number of positive solutions of (6.5.9) is equal to the number of roots of f in I_0. Note that the function f has no poles in I_0, thus by Rolle's theorem applied to f and f', we have

$$\sharp\{x \in I_0 \mid f(x) = 1\} \leq \sharp\{x \in I_0 \mid f''(x) = 0\} + 2.$$

Since

$$f''(x) = -\mathrm{coef}(a_3)x^{\alpha_3 - 2}(1-x)^{\beta_3 - 2}H_3(x) + \mathrm{coef}(b_4)x^{\alpha_4 - 2}(1-x)^{\beta_4 - 2}H_4(x),$$

where H_3 and H_4 are polynomials of degree at most two, we get
$f''(x) = 0 \Leftrightarrow \phi(x) = 1$, where

$$\phi(x) := -\frac{\mathrm{coef}(a_3)}{\mathrm{coef}(b_4)} \cdot \frac{x^{\alpha_3 - \alpha_4}(1-x)^{\beta_3 - \beta_4} H_3(x)}{H_4(x)}.$$

Thus applying Theorem 4.2 of Chapter 4 (with $\max(\deg H_3, \deg H_4) = 2$) we get $\sharp\{x \in I_0 \mid f''(x) = 0\} \leq 6$, and therefore (6.5.9) has at most eight positive solutions. □

6.5.2 Construction: second part of Theorem 6.48

In this subsection, we prove the following result

Proposition 6.54. *There exists a system* (6.5.2) *having seven non-degenerate positive solutions.*

In what follows, we impose $\alpha = \gamma_2 < \gamma_0$ to construct a system (6.5.2) with seven positive solutions (see Table 6.2). Assume that $\mathring{\mathfrak{C}}_0$ contains the valuation of one (which is the maximum possible for this case) positive solution of (6.5.2). Then, the lower hull Γ_0 has only one edge with negative slope, and thus both n_3 and n_4 are positive (see Figure 6.14 on the left).

Therefore, since $\alpha > 0$, both T_1 and T_2 do not have a vertex in L_0 (see Figure 6.18 for example). Consider the reduced system

$$\begin{array}{rcrcrcl} -1 & + & y_1^{m_1} & + & y_1^{m_2}y_2^{n_2} & = & 0, \\ \mathrm{coef}(b_4)y_1^{m_4}y_2^{n_4} & - & \mathrm{coef}(a_3)y_1^{m_3}y_2^{n_3} & + & \mathrm{coef}(c_2)y_1^{m_2}y_2^{n_2} & = & 0, \end{array} \tag{6.5.11}$$

of (6.5.5) with respect to v_0.

Lemma 6.55. *If the curves T_1 and T_2 intersect non-transversally at a point $v \neq v_0$ of type (III), then* (6.5.11) *does not have five positive solutions.*

Proof. Assume that T_1 and T_2 intersect non-transversally at a point v of type (III). We consider the case where $v \in \mathring{\mathsf{L}}_2$ since the other cases are symmetric. Then, since v is a common vertex of T_1 and T_2, we have

$$\frac{\alpha n_2}{m_3n_2 - m_2n_3} = \frac{\alpha n_2}{m_4n_2 - m_2n_4},$$

from which we deduce $(m_4-m_3)n_2-m_2(n_4-n_3) = 0$. This means that the segments $[(0,0),(m_2,n_2)]$ and $[(m_3,n_3),(m_4,n_4)]$ are parallel. Note that the Newton polytopes of the first and second equations of (6.5.11) are the triangles

$$[(0,0),(m_1,0),(m_2,n_2)] \quad \text{and} \quad [(m_2,n_2),(m_3,n_3),(m_4,n_4)]$$

respectively. Since $(m_4-m_3)n_2-m_2(n_4-n_3) = 0$, the vector $F_{0,2}$, normal to the facet $[(0,0),(m_2,n_2)]$ of $[(0,0),(m_1,0),(m_2,n_2)]$, is equal (up to a scalar multiplication) to the vector $F_{3,4}$, normal to the facet $[(m_3,n_3),(m_4,n_4)]$ of $[(m_2,n_2),(m_3,n_3),(m_4,n_4)]$. Therefore, the triangles

$$[(0,0),(m_1,0),(m_2,n_2)] \quad \text{and} \quad [(m_2,n_2),(m_3,n_3),(m_4,n_4)]$$

would alternate (see Definition 4.30 in Chapter 4), and thus by Theorem 4.3 of Chapter 4, the system (6.5.11) cannot reach the maximal number *five* of positive solutions. □

We assume in what follows that T_1 and T_2 do not intersect non-transversally at a point of type (III) belonging to the relative interior of a 1-cone of $\mathcal{E}$.

Remark 6.56. *The set $\mathfrak{D} = T_1 \cap T_2 \setminus (\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_1 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\})$ consists of transversal intersection points (which has cardinality at most 1 by Lemma 6.50) together with non-transversal points of type (II).*

Since intersection points of type (II) are not valuations of non-degenerate positive solutions of (6.5.2), Remark 6.56 shows that (6.5.2) has at most one non-degenerate positive solution with valuation in $\mathfrak{D}$, that is, by Lemma 6.50, a transversal point. Therefore, Table 6.2 shows that since $\alpha = \gamma_2 < \gamma_0$, the curves T_1 and T_2 intersect transversally at a point p.

We start our construction by finding a system (6.5.11) that has five positive solutions. Since systems of two trinomials in two variables having five positive solutions are hard to generate (c.f. [DRR07]), we will borrow one from the literature and base our construction upon it.

First, we define a univariate function f such that for some constant c, the equation $f = c$ has the same number of solutions in $]0,1[$ as that of positive solutions of (6.5.11). Assume without loss of generality that $\mathrm{coef}(a_3) = -1$. The first equation of (6.5.11) may be written as $y_2 = x^k(1-x)^l$, where $x := y_1^{m_1}$, $k = -m_2/(m_1n_2)$ and $l = 1/n_2$. It is clear that $y_1, y_2 > 0 \Leftrightarrow x \in I_0 :=]0,1[$. Since we are looking for solutions of (6.5.11) with non-zero coordinates, we divide its second equation by $y_1^{m_2}y_2^{n_2}$. Plugging y_1 and y_2 in the second equation of 6.5.11, we get

$$\operatorname{coef}(c_2) + x^{k_3}(1-x)^{l_3} + \operatorname{coef}(b_4)x^{k_4}(1-x)^{l_4} = 0, \tag{6.5.12}$$

where $k_i = \dfrac{m_i n_2 - m_2 n_i}{m_1 n_2}$ and $l_i = \dfrac{n_i - n_2}{n_2}$ for $i = 3, 4$. The number of positive solutions of (6.5.11) is equal to the number of solutions of (6.5.12) in I_0. Therefore we want to compute values of $\operatorname{coef}(c_2)$, $\operatorname{coef}(b_4)$ and (m_i, n_i) for $i = 1, 2, 3, 4$ such that $f(x) = -\operatorname{coef}(c_2)$ has five solutions in I_0, where

$$f(x) := x^{k_3}(1-x)^{l_3} + \operatorname{coef}(b_4) \cdot x^{k_2}(1-x)^{l_2}. \tag{6.5.13}$$

Note that the function f has no poles in I_0, thus by Rolle's theorem we have $\sharp\{x \in I_0 \mid f(x) = 1\} \leq \sharp\{x \in I_0 \mid f'(x) = 0\} + 1$. Since

$$f'(x) = x^{k_3-1}(1-x)^{l_3-1}\rho_3(x) + a_4 x^{k_4-1}(1-x)^{l_4-1}\rho_4(x),$$

where $\rho_i(x) = k_i - (k_i + l_i)x$ for $i = 3, 4$, we get $f'(x) = 0 \Leftrightarrow \phi(x) = 1$, where

$$\phi(x) := -\operatorname{coef}(b_4)\frac{x^{k_4-k_3}(1-x)^{l_4-l_3}\rho_4(x)}{\rho_3(x)}. \tag{6.5.14}$$

Consider the system

$$x^6 + (44/31)y^3 - y = y^6 + (44/31)x^3 - x = 0, \tag{6.5.15}$$

taken from [DRR07], which has five positive solutions. The rational function (6.5.14), associated to (6.5.15) is

$$\phi_0(x) = (44/31)^{5/6} \cdot \frac{x^{1/6}(1-x)^{1/3}(-11/4 + 9x/4)}{(-35/12 + 11x/4)}.$$

Thus, if

$$\begin{aligned} &\operatorname{coef}(b_4) = -\left(\frac{44}{31}\right)^{\frac{5}{6}}, \quad k_4 - k_3 = \frac{1}{6}, \quad l_4 - l_3 = \frac{1}{3}, \\ &k_4 = -\frac{11}{4} \qquad\qquad \text{and} \qquad\qquad k_3 = -\frac{35}{12}, \end{aligned} \tag{6.5.16}$$

then $\phi(x) = 1$ has four positive solutions in I_0. Assume that equalities in (6.5.16) hold true. Plotting the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$, we get that the graph of f has four critical points contained in I_0 with critical values situated below the x-axis. Moreover, this graph intersects transversally the line $\{y = -0.36008\}$ in five points with the first coordinates belonging to I_0. Therefore, the equation $f(x) = -0.36008$ has five non-degenerate positive solutions in I_0. In what follows, we find $(m_i, n_i) \in \mathbb{Z}^2$ for $i = 1, 2, 3, 4$, satisfying the equalities in (6.5.16) so that (6.5.11) has five non-degenerate positive solutions.

Assume that $m_2 > 0$ and recall that both m_1 and n_2 are positive. The equalities in (6.5.16) show that $l_i > 0$, $k_i < 0$ and $k_i < l_i$ for $i = 3, 4$, therefore we have $0 < n_2 < n_i$, $m_i n_2 - n_i m_2 < 0$ and $(m_i - m_1)n_2 - n_i(m_2 - m_1) < 0$ for $i = 3, 4$. Plotting the three points $(0, 0)$, $(m_1, 0)$ and

(m_2, n_2), we deduce from the latter inequalities that the points (m_3, n_3) and (m_4, n_4) belong to the region B_1 of Figure 6.17.

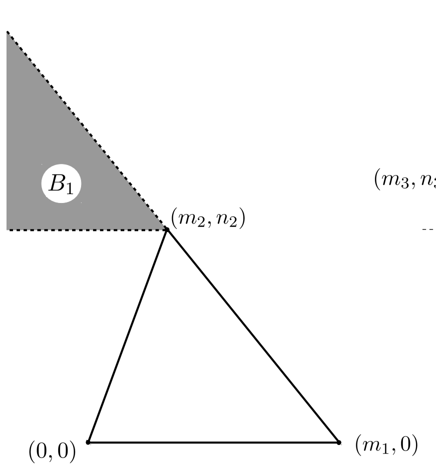

Figure 6.17: The region B_1 and triangle $B_{1,1}$

We also deduce from equalities in (6.5.16) that $l_4 > l_3$ and $k_4 > k_3$, and thus $n_4 > n_3$ and $(m_4 - m_3)n_2 - (n_4 - n_3)m_2 > 0$. Fixing (m_3, n_3) in the region B_1, we obtain that (m_4, n_4) belongs to the triangle $B_{1,1}$ depicted in Figure 6.17.

Note that the vertex $v_1 \in \mathsf{L}_2$ (resp. $v_2 \in \mathsf{L}_2$) of T_1 (resp. T_2) has coordinates

$$\frac{\alpha}{m_3 n_2 - n_3 m_2}(n_2, -m_2) \quad \left(\textit{resp.} \quad \frac{\alpha}{m_4 n_2 - n_4 m_2}(n_2, -m_2) \right),$$

and thus from $m_3 n_2 - n_3 m_2 < m_4 n_2 - n_4 m_2 < 0$, we deduce that the first coordinate of v_2 is smaller than that of v_1 (see Figure 6.18).

All these restrictions impose that there exists a transversal intersection point of T_1 and T_2 in C_2 (see Figure 6.18 for example). Moreover, since $\operatorname{coef}(b_4) < 0$ (see (6.5.16)), $\operatorname{coef}(a_3) = -1$ (by assumption) and $\operatorname{coef}(a_0) = \operatorname{coef}(b_0) = -1$, Proposition 6.27 shows that the intersection point p is the valuation of a positive solution of (6.5.2). Since $\operatorname{coef}(c_2) = 0.36008$ (from the choice $f(x) = -\operatorname{coef}(c_2) = -0.36008$), for any negative $\operatorname{coef}(c_0)$, the facial subpolynomial $\operatorname{coef}(c_0) + 0.36008y^{n_2}$ of $f_{0,t}$ has a positive root. We choose $\operatorname{coef}(c_0)$ to be equal to -0.36008 so that the root for $-0.36008 + 0.36008y^{n_2}$ becomes equal to 1.

According to this analysis, it suffices to choose exponents and coefficients of (6.5.2) satisfying $m_1 = 6$, $(m_2, n_2) = (3, 6)$, $(m_3, n_3) = (-14, 7)$, $(m_4, n_4) = (-12, 9)$, $a_0 = -1$, $a_2 = 1$, $a_3 = -t^\alpha$, $b_0 = -1 + 0.36008t^{\gamma_0}$ (verifying $\gamma_0 > \alpha$), $b_2 = -1 + t^\alpha$ and $b_4 = -(44/31)^{5/6}t^\alpha$. Therefore, the system

$$\begin{array}{rllll} -1 & + y_1^6 & + y_1^3 y_2^6 & - t^\alpha y_1^{-14} y_2^7 & = 0, \\ -1 + 0.36008t^{\gamma_0} & + y_1^6 & + (1 - 0.36008t^\alpha) y_1^3 y_2^6 & - (44/31)^{\frac{5}{6}} t^\alpha y_1^{-12} y_2^9 & = 0, \end{array} \tag{6.5.17}$$

which has seven non-degenerate solutions, proves Proposition 6.54.

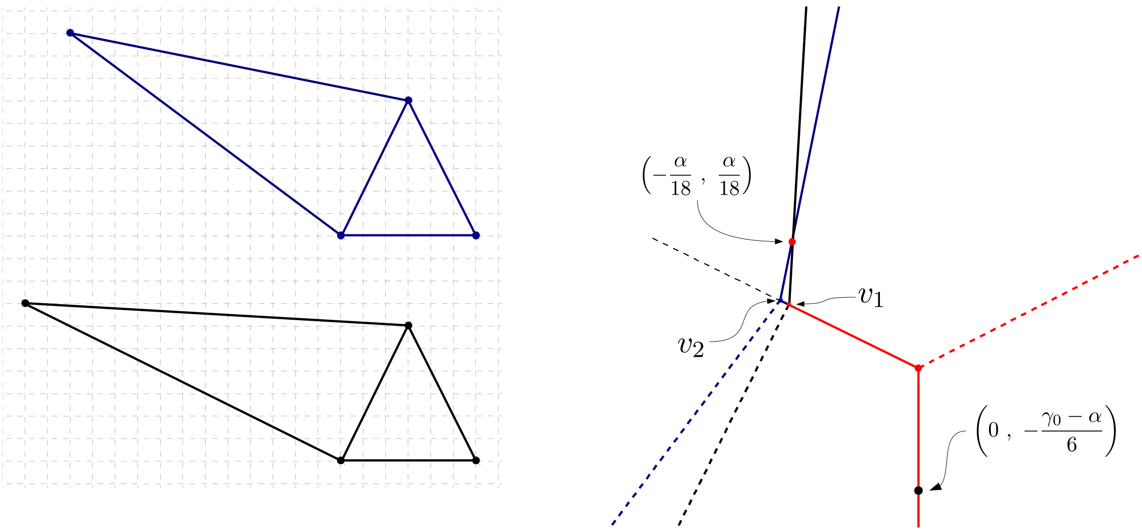

Figure 6.18: Newton polytopes and tropical curves associated to a normalized system having seven positive solutions.

6.5.2.1 A software computation

Using Maple 17 as well as the libraries FGb and RS, Pierre-Jean Spaenlehauer [Spa] provided us with a computation he made of the non-degenerate positive solutions of a system (6.5.17) for $\gamma_0 = 7$ and $\alpha = 1$ that goes as follows. For computational reasons, he has replaced the real number $(44/31)^{5/6}$ in (6.5.17) by the fraction

$$\frac{26807502408507435267952730104920543812845885439976}{20022295568917288472920446333489413342983920443429}$$

which approximates $(44/31)^{5/6}$. For $t = 1/100\,000$, the computer software has found seven positive solutions. An approximation of these solutions goes as follows.

$$(0.99999,\ 0.00001),\ (0.99171,\ 0.60681),\ (0.96651,\ 0.76771),\ (0.95765,\ 0.79907),$$

$$(0.95201,\ 0.81642),\ (0.88602,\ 0.95151),\ (0.53645,\ 1.61099).$$

6.6 Proof of Theorem 6.3 (part 1).

Consider a highly non-degenerate normalized system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0. \end{aligned} \tag{6.6.1}$$

satisfying that all a_i and b_j are in $\mathbb{RK}^*$ and verify $\mathrm{ord}(a_i) = \mathrm{ord}(b_j) = 0$, all w_i are in $\mathbb{Z}^2$, both m_1, n_2 are positive and both α, β are real numbers. This Section is devoted to proving the following result.

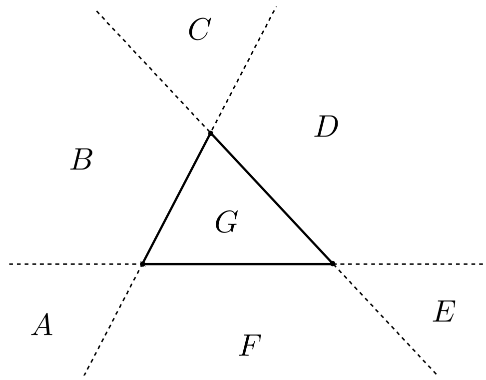

Figure 6.19: The seven regions.

Theorem 6.57. *If* $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ *for* $i = 0, 2$ *and either* $\alpha \neq \beta$ *or* $\alpha < 0$*, then the sharp bound on the number of non-degenerate positive solutions of* (6.6.1) *is six.*

The system (6.4.12) appearing in Example 6.38 of Section 6.4, satisfies the hypotheses of Theorem 6.57 and has six non-degenerate positive solutions. Therefore, if Theorem 6.57 holds true, then six is a sharp bound on the number of non-degenerate positive solutions of (6.6.1) .

In what follows, we assume the hypotheses of Theorem 6.57. As in the previous section, v_0 denotes the origin of $\mathcal{E}$. Let Δ_1 and Δ_2 (resp. τ_1 and τ_2, T_1 and T_2) denote the Newton polytopes (resp. dual subdivisions, tropical curves) associated to the first and second equations respectively.

It follows from Corollary 6.40 that since $\mathrm{coef}(a_i) = \mathrm{coef}(b_i)$ for $i = 0, 2$, the system (6.6.1) does not have a positive solution with valuation at a non-transversal intersection point of type (II).

We now show why Theorem 6.57 is trivial if both $\mathrm{coef}(a_0)$ and $\mathrm{coef}(a_2)$ are positive. Note that the reduced system of (6.6.1) with respect to v_0 will not have positive solutions, and if T_1 and T_2 intersect non-transversally at a cell of type (I), such a cell does not contain the valuations of positive solutions of (6.6.1). Moreover, Theorem 6.15 in Section 6.3 shows that (6.6.1) has at most three positive solutions with valuations transversal intersection points of T_1 and T_2. Therefore, if there does not exist a non-transversal intersection point of type (III) in the relative interior of a 1-cone of $\mathcal{E}$, then (6.6.1) has at most three positive solutions. Otherwise, if there exists a non-transversal intersection point $v \neq v_0$ of type (III), then Remark 6.42 and Lemma 6.44 in Section 6.4 show that (6.6.1) has at most three positive solutions.

Using similar arguments as in Section 6.5, in what follows we assume that

$$\mathrm{coef}(a_0) = -1 \quad \text{and} \quad \mathrm{coef}(a_2) = 1.$$

Therefore, Lemma 6.33 in Section 6.4 shows that if there exists a non-transversal cell $\mathfrak{C}_1$ of type (I) contained in L_1, then $\mathring{\mathfrak{C}}_1$ does not contain valuations of positive solutions of (6.6.1). In this section, the only cells of $T_1 \cap T_2$ that may contain valuations of non-degenerate positive solutions of (6.6.1) are the following.

- Non-transversal cells of type (I) contained in $\mathsf{L}_0 \cup \mathsf{L}_2$.
- Transversal intersection points in $\cup_{i=0}^{2} \mathring{\mathsf{C}}_i$.
- A non-transversal intersection point of type (III) contained in $\mathring{\mathsf{L}}_0 \cup \mathring{\mathsf{L}}_1$.

The reason we omit the case where there could be an intersection point v of type (III) in $\mathring{\mathsf{L}}_2$ is the following. Assume that T_1 and T_2 intersect non-transversally at a point $v \in \mathring{\mathsf{L}}_2$ of type (III). Then, since v is the intersection of a vertex in $\mathring{\mathsf{L}}_2$ of T_1 and a vertex of T_2 in the same 1-cone of $\mathcal{E}$, we have $\alpha/(m_3n_2 - m_2n_3) = \beta/(m_4n_2 - m_2n_4)$. Moreover, since T_1 and T_2 do not intersect non-transversally at a point of type (III) belonging to $\mathring{\mathsf{L}}_0$ (see Lemma 6.43), we have $\alpha/n_3 \neq \beta/n_4$. The highly non-degenerate normalized system

$$\begin{aligned} c_0 + z_1^{k_1} + c_2 z_1^{k_2} z_2^{l_2} + c_3 t^{\alpha} z_1^{k_3} z_2^{l_3} &= 0, \\ d_0 + z_1^{k_1} + d_2 z_1^{k_2} z_2^{l_2} + d_4 t^{\beta} z_1^{k_4} z_2^{l_4} &= 0, \end{aligned} \tag{6.6.2}$$

where $\mathrm{coef}(c_0) = \mathrm{coef}(d_0) = -1$ and $\mathrm{coef}(c_2) = \mathrm{coef}(d_2) = 1$, has the same number of non-degenerate positive solutions as (6.6.1), and the associated tropical curves $\tilde{T}_1$ and $\tilde{T}_2$ intersect at a point $\tilde{v}$ of type (III) contained in L_0. Indeed, divide the first and the second equations of (6.6.1) by a_2 and b_2 respectively, and make the monomial coordinate change $(y_1, y_2) \mapsto (z_1, z_2)$ such that $y_1^{m_1} = z_1^{k_2} z_2^{l_2}$ and $y_1^{m_2} y_2^{n_2} = z_1^{k_1}$ for some integers $k_1 > 0$, k_2 and $l_2 > 0$. One can easily check that $\alpha/l_3 = \beta/l_4$, and thus $\tilde{T}_1$ and $\tilde{T}_2$ intersect non-transversally at a point of type (III) contained in $\mathring{\mathsf{L}}_0$. Moreover, since (6.6.2) is also highly non-degenerate, we get that $\mathring{\mathsf{L}}_1 \cup \mathring{\mathsf{L}}_2$ does not contain non-transversal intersection points of type (III).

6.6.1 First case: $0 < \alpha < \beta$

The tropical curves T_1 and T_2 intersect non-transversally at the origin v_0 of $\mathcal{E}$ and at three linear components of type (I) denoted by $\mathfrak{C}_i$ for $i = 0, 1, 2$ such that $\mathfrak{C}_i \subset \mathsf{L}_i$.

Recall that by Lemma 6.34 in Section 6.4, the polynomials

$$f_{0,t} = \mathrm{coef}(c_0)t^{\gamma_0} + \mathrm{coef}(c_2)t^{\gamma_2} y^{n_2} - \mathrm{coef}(a_3)t^{\alpha} y^{n_3} + \mathrm{coef}(b_4)t^{\beta} y^{n_4}$$

$$\text{and} \quad f_{2,t} := ct^{\delta} - \mathrm{coef}(a_3)t^{\alpha} y^{\frac{m_3n_2 - m_2n_3}{n_2}} + \mathrm{coef}(b_4)t^{\beta} y^{\frac{m_4n_2 - m_2n_4}{n_2}},$$

where $c_i := b_i - a_i$, $\gamma_i := \mathrm{ord}(c_i)$ for $i = 0, 2$, $c := \mathrm{coef}(c_2 - c_0)$ and $\delta := \mathrm{ord}(c_2 - c_0)$, are approximation polynomials of (6.6.1) for $\mathfrak{C}_0$ and $\mathfrak{C}_2$ respectively.

6.6.1.1 There exists a non-transversal intersection of type (III)

Here, we study the case where T_1 and T_2 intersect non-transversally at a point v of type (III) contained in $\mathring{\mathsf{L}}_0 \cup \mathring{\mathsf{L}}_1$. Note that if $v \in \mathring{\mathsf{L}}_i$ for some $i = 0, 1$, then the vertices v and v_0 are endpoints of $\mathfrak{C}_i$. Let $\mathfrak{C} \subset T_1 \cap T_2$ denote the intersection component $\mathfrak{C}_0 \cup \mathfrak{C}_2 \cup \{v\} \cup \{v_0\}$.

Lemma 6.44 shows that (6.6.1) has at most *one* non-degenerate positive solution with valuation a transversal intersection point of T_1 and T_2. We want to prove the following result.

Proposition 6.58. *The system* (6.6.1) *has at most six non-degenerate positive solutions with valuation in* $\mathfrak{C}$. *Moreover, if* (6.6.1) *has six non-degenerate positive solutions with valuation in* $\mathfrak{C}$, *then* (6.6.1) *does not have a positive solution with valuation a transversal intersection point of* T_1 *and* T_2.

Since there exists a non-transversal intersection of type (III), Theorem 6.57 becomes a consequence of Proposition 6.58 given that the latter holds true.

• **First case: $v \in \mathsf{L}_0$.** We have $n_4 < n_3 < 0$. Indeed, the intersection point v belongs to L_0 and satisfies $v = (0, \alpha/n_3) = (0, \beta/n_4)$ (since v is a common vertex of T_1 and of T_2). Therefore, we get $\beta/n_4 = \alpha/n_3 < 0$, and thus from $0 < \alpha < \beta$, we get $n_4 < n_3 < 0$.

Recall that Γ_0 (resp. Γ_2) is the lower part of the convex hull of points in

$$\{(0, \gamma_0),\ (n_2, \gamma_2),\ (n_3, \alpha),\ (n_4, \beta),\ \}$$

$$(\text{resp.}\quad \{(0, \delta),\ ((m_3 n_2 - m_2 n_3)/n_2, \alpha)\,,\ ((m_4 n_2 - m_2 n_4)/n_2, \beta)\})\,.$$

Since $n_4 < n_3 < 0 < n_2$ and $\alpha, \beta, \gamma_0, \gamma_2 > 0$, the lower hull Γ_0 contains an edge $e_1 \subset \Gamma_0$ with endpoints (n_4, β) and (n_3, α), where e_1 has negative slope (see Figure 6.20 for example). Moreover, from $\alpha/n_3 = \beta/n_4$, we deduce that the facial subpolynomial $f_0^{(1)}(y) = -\operatorname{coef}(a_3) y^{n_3} + \operatorname{coef}(b_4) y^{n_4}$ (which is associated to e_1) is obtained from $f_{0,t}(t^{-\lambda_1} y)/t^{\mu_1}$, where $\lambda_1 = \beta/n_4$ and $\mu_1 = 0$. Therefore, by Corollary 6.12 of Section 6.2, if $f_0^{(1)}$ has a positive root, it does not correspond to a positive non-degenerate solution of (6.6.1) with valuation in $\mathring{\mathfrak{C}}_0$. Therefore, $\mathring{\mathfrak{C}}_0$ contains the valuations of at most *two* positive solutions of (6.6.1). Note that by Remark 6.42 of Section 6.4, the intersection point v is the valuation of at most *two* non-degenerate positive solutions of (6.6.1), and recall that by Remark 6.35 of Section 6.4, we have $\mathring{\mathfrak{C}}_2$ contains the valuation of at most *two* positive solutions.

From Subsection 6.4.3, the number of positive solutions of (6.6.1) with valuation v_0 is equal to the number of positive solutions of the reduced system of

$$\begin{aligned} -1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ c_0 t^{\gamma_0} + c_2 t^{\gamma_2} y_1^{m_2} y_2^{n_2} - a_3 t^{\beta} y_1^{m_3} y_2^{n_3} + b_4 t^{\alpha} y_1^{m_4} y_2^{n_4} &= 0 \end{aligned} \tag{6.6.3}$$

with respect to v_0, with $c_i t^{\gamma_i} = b_i - a_i$, $\operatorname{ord}(c_i) = 0$ and $\gamma_i \geq 0$ for $i = 0, 2$.

We prove Proposition 6.58 by analyzing the different cases for the system (6.6.3). Recall Corollary 6.12 and that by an *edge* of Γ_0 and Γ_2, we mean a line segment of these lower hulls supporting only a binomial.

i) Assume that there exists only one element of the set $\{\alpha, \gamma_0, \gamma_2\}$ that is equal to $\min(\alpha, \gamma_0, \gamma_2)$. Recall that the reduced system of (6.4.14) with respect to v_0 has no real positive solutions. If $\mathring{\mathfrak{C}}_0$, $\mathring{\mathfrak{C}}_2$ or $\{v\}$ contains the valuations of at most *one* positive solution, then $\mathfrak{C}$ contains the valuations of at most *five*, and we are done.

Assume that (6.6.1) has two non-degenerate positive solutions with valuations in each of $\mathring{\mathfrak{C}}_0$, $\mathring{\mathfrak{C}}_2$ and $\{v\}$. Note that since there exist positive solutions with valuation v, the system (6.4.24) from Subsection 6.4.4 shows that $\operatorname{coef}(a_3)\operatorname{coef}(b_4) > 0$. The two positive roots of $f_{0,t}$ (which are associated to two positive solutions of (6.6.1) with valuation in $\mathring{\mathfrak{C}}_0$) correspond to two edges of $\Gamma_0 \setminus \{e_1\}$ with negative slopes. Since $n_4 < n_3 < 0 < n_2$, we have $\beta > \alpha > \gamma_0 > \gamma_2$ (see Figure 6.20 on the left), and by Descartes' rule of sign, we get $\operatorname{coef}(c_0)\operatorname{coef}(a_3) > 0$ and $\operatorname{coef}(c_2)\operatorname{coef}(c_0) < 0$, thus $\operatorname{coef}(c_2)\operatorname{coef}(a_3) < 0$. Similarly, since $0 < \delta < \alpha < \beta$ and (6.6.1) has two positive solutions with valuations in $\mathring{\mathfrak{C}}_2$, applying Corollary 6.12 on $f_{2,t}$, we deduce that $m_4 n_2 - m_2 n_4 < m_3 n_2 - m_2 n_3 < 0$ (see Figure 6.20 on the right). Moreover, since $\delta = \min(\gamma_0, \gamma_2) = \gamma_2$, the coefficient c, appearing in $f_{2,t}$, has the same sign as that of $\operatorname{coef}(c_2)$. Therefore by Descarte's rule of sign, the number of sign changes of $f_{2,t}$ is equal to one, thus a contradiction to (6.6.1) having two non-degenerate positive solutions with valuations in $\mathring{\mathfrak{C}}_2$. We deduce that (6.6.1) has at most *five* positive solutions with valuation in $\mathfrak{C}$.

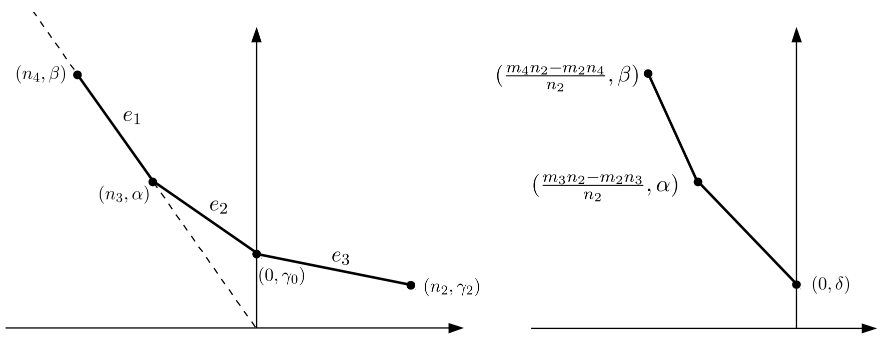

Figure 6.20: Examples of graphs Γ_0 and Γ_2 for $n_4 < n_3 < 0 < n_2$ and $m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0$.

ii) Assume that $\gamma_0 = \gamma_2 < \alpha$. Recall that the reduced system of (6.6.1) with respect to v_0 has at most *one* positive solution. Moreover, the lower hull Γ_0 contains two edges e_1 and e_2 (corresponding to the facial subpolynomials $-\operatorname{coef}(a_3)y^{n_3} + \operatorname{coef}(b_4)y^{n_4}$ and $\operatorname{coef}(b_4)y^{n_4} + \operatorname{coef}(c_0)$ respectively) with negative slope, and a horizontal edge e_3 corresponding to $\operatorname{coef}(c_0) + \operatorname{coef}(c_2)y^{n_2}$ (see Figure 6.21). Therefore, only e_2 may correspond to a positive solution of (6.6.1) with valuation in $\mathring{\mathfrak{C}}_0$, and thus $\mathfrak{C}$ contains the valuation of at most *six* positive solutions.

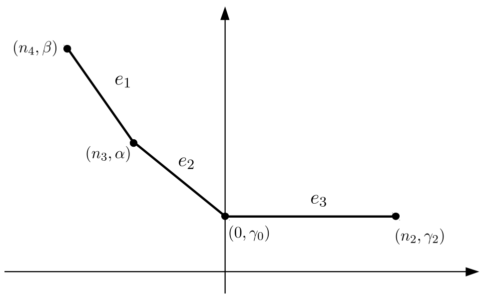

Figure 6.21: An example of Γ_0 for $\gamma_0 = \gamma_2 < \alpha$.

Assume that this bound is reached. We prove that (6.6.1) does not have a non-degenerate positive solution with valuation a transversal intersection point of T_1 and T_2. Recall that $\delta \geq \gamma_0$. We have $\delta = \gamma_0$. Indeed, if $\delta > \gamma_0$, then $\operatorname{coef}(c_0) = -\operatorname{coef}(c_2)$, and the reduced system of (6.6.1) with respect to v_0 may be written as

$$-1 + y_1^{w_1} + y_1^{m_2}y_2^{n_2} = -1 + y_1^{m_2}y_2^{n_2} = 0, \tag{6.6.4}$$

which does not have positive solutions. This is a contradiction to (6.6.1) having six positive solutions with valuation in $\mathfrak{C}$.

Since $\mathring{\mathfrak{C}}_2$ contains the valuations of two positive solutions of (6.6.1) (by assumption), all edges of Γ_2 have negative slope, and using similar arguments as in **i)**, we have

$$m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0. \tag{6.6.5}$$

The latter inequalities together with $n_4 < n_3 < 0$ show that the points (m_3, n_3) and (m_4, n_4) belong to the region A of Figure 6.19. Moreover, since both α and β are positive, each of T_1 and T_2 has a vertex v_1 and v_2 respectively in L_2. Lemma 6.44 shows that since $v \in \mathsf{L}_0$, the curves T_1 and T_2 intersect transversally in at most *one* point p.

Assume that such an intersection p exists, and that p is the valuation of a positive solution of (6.6.1), we prove that this gives a contradiction. Then by Lemma 6.44, we have $p \in \mathsf{C}_2$. Moreover, since $\mathrm{coef}(a_2) > 0$, we deduce from Proposition 6.27 that both $\mathrm{coef}(a_3)$ and $\mathrm{coef}(b_4)$ are negative. Descartes' rule of signs applied to the polynomial (6.4.25) of Subsection 6.4.4 associated to the reduced system with respect to v shows that

$$\frac{m_3n_4 - m_4n_3}{n_4 - n_3} > m_1 > 0.$$

Indeed, since (6.4.25) has two positive solutions and $m_1 > 0$. Therefore, from $n_4 < n_3$ we get $m_3n_4 - m_4n_3 < 0$, and thus comparing the coordinates of v_1 to those of v_2 using the inequalities in (6.6.5) gives that the first coordinate of v_1 is smaller than that of v_2 (See Figure 6.22 on the right). Moreover, the inequality $m_3n_4 - m_4n_3 < 0$ shows that fixing (m_3, n_3) in the region A of Figure 6.19, the point (m_4, n_4) is contained in region A_1 of Figure 6.22. However, under these constraints on (m_3, n_3), (m_4, n_4), v_1 and v_2, the tropical curves T_1 and T_2 do not intersect transversally at a point contained in the 2-cone C_2, a contradiction.

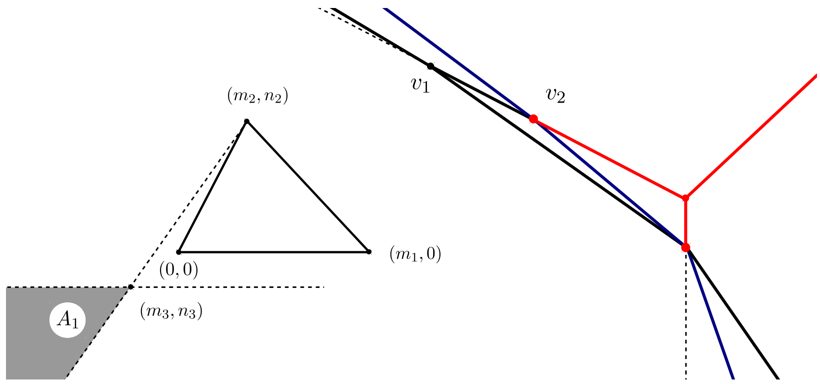

Figure 6.22: The region A_1 with respect to the triangle $[(0, 0), (m_1, 0), (m_2, n_2)]$.

iii) Assume that $\alpha = \gamma_0 < \min(\gamma_2, \beta)$ (we omit the case where $\alpha = \gamma_2 \leq \beta < \gamma_0$ since it is similar). Recall that the reduced system (6.4.16) with respect to v_0 has at most *two* positive solutions. The only edge of Γ_0 having a negative slope is e_1, thus $\mathring{\mathfrak{C}}_0$ does not contain valuations of positive solutions of (6.6.1) (see Figure 6.23 left). Moreover, since $\delta = \gamma_0 = \alpha$, the lower hull Γ_2 contains at most *one* edge with negative slope (see Figure 6.23 right). Therefore, there exists at most *five* solutions of (6.6.1) with valuation in $\mathfrak{C}$.

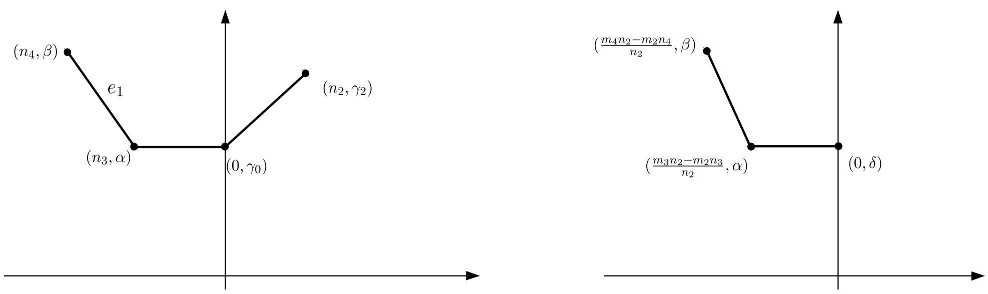

Figure 6.23: Examples of Γ_0 and Γ_2 for $\gamma_0 = \alpha$.

iv) Assume that $\alpha = \gamma_0 = \gamma_2 < \beta$. Recall that the reduced system (6.4.18) with respect to v_0 has at most *three* positive solutions. The lower hull Γ_0 contains only e_1 and one horizontal edge (See Figure 6.24 on the left), and thus $\mathring{\mathfrak{C}}_0$ does not contain valuations of positive solutions of (6.6.1). Recall that by Lemma 6.44, since $v \in \mathsf{L}_0$, if T_1 and T_2 intersect transversally, then this transversal intersection point belongs to C_2. Note that since $\alpha > 0$, if T_1 does not have a vertex in $\mathring{\mathsf{L}}_1 \cup \mathring{\mathsf{L}}_2$, then T_1 does not have an edge contained in C_2, and thus T_1 and T_2 do not intersect transversally at a point in C_2. The number of edges of Γ_0 with negative slope depends on whether δ is equal to γ_0 or not. We distinguish two cases for δ and deduce that if (6.6.1) has six positive solutions with valuation in $\mathfrak{C}$, then T_1 does not have a vertex in either L_1 or L_2.

Assume first that $\delta = \gamma_0$. We deduce from $f_{2,t}$ that the lower hull Γ_2 contains one horizontal edge and at most *one* other edge with non-zero slope (see Figure 6.24 on the center). Therefore (6.6.1) has at most *one* positive solution with valuation in $\mathring{\mathfrak{C}}_2$. This means that the maximal number of positive solutions of (6.6.1) with valuations in the intersection component $\mathfrak{C}$ is equal to six. Assuming that this bound is reached, we get that the reduced system (6.4.18) with respect to v_0 has the maximum of three positive solutions. Therefore, since such a system is supported on a circuit, its support $\mathcal{W}_0 := \{(0,0),\ (m_1,0),\ (m_2,n_2),\ (m_3,n_3)\}$, satisfies the following. The triangle Δ_w, formed by any three distinct points of $\mathcal{W}_0$ does not contain the remaining forth point of $\mathcal{W}_0$. Since $n_3 < 0$, the latter restrictions mean that (m_3,n_3) is contained in region F of Figure 6.19. Therefore, since $\alpha > 0$, the tropical curve T_1 does not have a vertex in $\mathring{\mathsf{L}}_1 \cup \mathring{\mathsf{L}}_2$ (see Figure 6.24 on the right), and thus no transversal intersection points.

Assume now that $\delta > \gamma_0$. Then (6.6.1) may have two positive solutions with valuation in $\mathring{\mathfrak{C}}_2$. Moreover, if this bound is reached, then $m_3n_2 - m_2n_3 > \min(0, m_4n_2 - m_2n_4)$. Indeed, since otherwise Γ_2 will not be optimally sloped (c.f. Figure 6.25 for example).

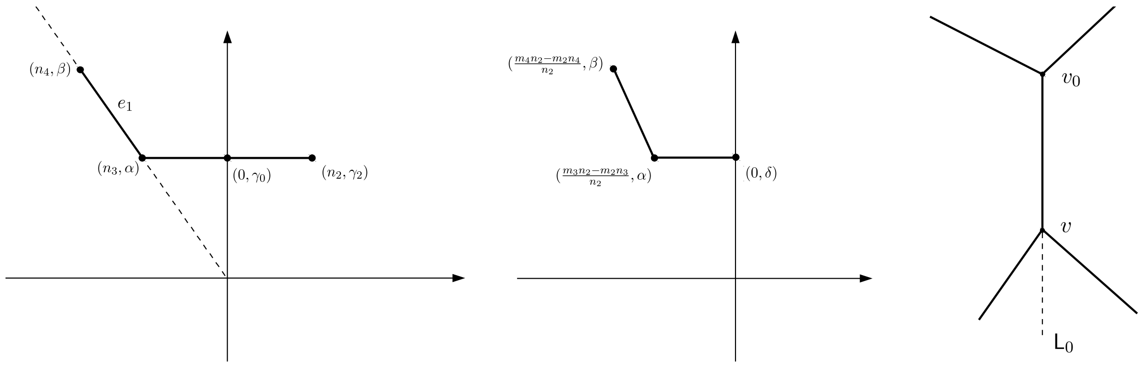

Figure 6.24: From left to right: Γ_0, Γ_2 and T_1 for $\alpha = \gamma_0 = \gamma_2 < \beta$.

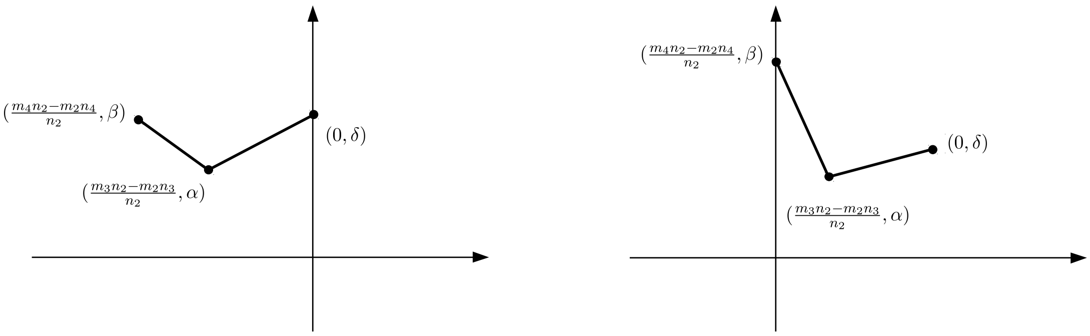

Figure 6.25: Examples where Γ_2 is not optimally sloped for $\alpha = \gamma_0 = \gamma_2 < \beta$.

Note that $\delta > \gamma_0$ means that we have $\operatorname{coef}(c_0) = -\operatorname{coef}(c_2)$, and thus the reduced system

$$-1 + y_1^{m_2} y_2^{n_2} + y_1^{m_1} = \operatorname{coef}(c_0) + \operatorname{coef}(c_2) y_1^{m_2} y_2^{n_2} - \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} = 0$$

with respect to v_0 has at most *two* positive solutions. Moreover, a non-degenerate positive solution (α, β) of the latter system satisfies

$$-1 + \alpha^{m_1} + c_3^{n_2} \alpha^{(m_2 n_3 - n_2(m_3 - m_1))/n_3} = 0 \tag{6.6.6}$$

with $\operatorname{coef}(a_3) < 0$ and $c_3 = (-1/\operatorname{coef}(a_3))^{1/n_3}$. Since (6.6.1) has six positive solutions with valuation in $\mathfrak{C}$ (by assumption), each of $\{v_0\}$, $\{v\}$ and $\mathring{\mathfrak{E}}_2$ contains the valuations of at most two positive solutions. Moreover, since $m_1 > 0$, by Descartes' rule of signs applied to (6.6.6), we we have $(m_2 n_3 - n_2(m_3 - m_1))/n_3 < 0$, and thus $m_2 n_3 - n_2(m_3 - m_1) > 0$. The latter inequality together with $m_3 n_2 - m_2 n_3 > 0$ show that (m_3, n_3) belongs to the region F_1 represented in Figure 6.26. Therefore, since $\alpha > 0$, the tropical curve T_1 does not have a vertex in $\mathring{\mathsf{L}}_1 \cup \mathring{\mathsf{L}}_2$.

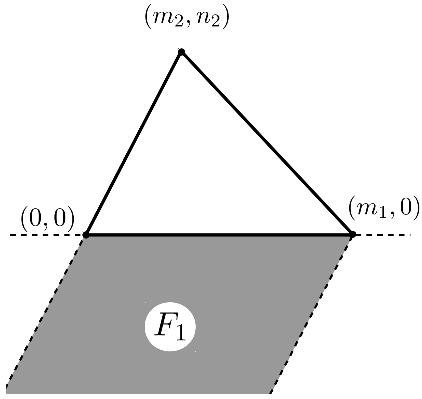

Figure 6.26: The region F_1.

This concludes the proof of Proposition 6.58 in the case where $v \in \mathsf{L}_0$.

• **Second case:** $\boldsymbol{v \in \mathsf{L}_1}$. Recall that the reduced system with respect to v is

$$y_1^{m_1} + y_1^{m_2} y_2^{n_2} + \operatorname{coef}(a_3) y_1^{m_3} y_2^{n_3} = y_1^{m_1} + y_1^{m_2} y_2^{n_2} + \operatorname{coef}(b_4) y_1^{m_4} y_2^{n_4} = 0. \qquad (6.6.7)$$

Note that this system has positive solutions only if each of $\operatorname{coef}(a_3)$ and $\operatorname{coef}(b_4)$ is negative. Similarly to the the case where $v \in \mathsf{L}_0$, we make a simple analysis on $f_{0,t}$, $f_{2,t}$ and on the reduced system of (6.6.3) with respect to v_0. This analysis is based on the inequalities between α, β, γ_0 and γ_2. The cases from **i)** to **iv)** are the same that been considered in the case where $v \in \mathsf{L}_0$. The entries in the following table represent the maximum number of positive solutions of (6.6.1) with valuation in the associated cell of $T_1 \cap T_2$.

Intersection Locus	**i)**	**ii)**	**iii)**	**iv)**
$\{v_0\}$	0	1	2	3 | 2
$\mathfrak{E}_0$	3	2	2	1 | 1
$\mathfrak{E}_2$	2	2	1	1 | 2

We deduce that (6.6.1) has at most *five* positive solutions with valuation in $\mathfrak{C} \setminus v$. Assume first that T_1 and T_2 intersect transversally at a point p and that p is the valuation of a positive solution of (6.6.1). Lemma 6.44 shows that $p \in \mathsf{C}_0$, thus from Proposition 6.27, we have that $\operatorname{coef}(a_3) > 0$ and $\operatorname{coef}(b_4) > 0$. Therefore (6.6.7) has no positive solutions, and consequently (6.6.1) has at most *five* positive solutions in $\mathfrak{C}$.

Assume now that (6.6.7) has two positive solutions (thus $\operatorname{coef}(a_3, \operatorname{coef}(b_4)) < 0$, and if T_1 and T_2 intersect transversally at p, it is not a valuation of a positive solution) and that the component $\mathfrak{C} \setminus \{v\}$ contains the valuations of five positive solutions. We prove that these assumptions give a contradiction. Since the system (6.6.1) has five positive solutions with valuations in $\mathfrak{C} \setminus \{v\}$, then Γ_0 and Γ_2 are both optimally sloped. Therefore, from $0 < \alpha < \beta$, we deduce the inequalities $n_4 < n_3$ and $m_4 n_2 - m_2 n_4 < m_3 n_2 - m_2 n_3$. Recall that the vertices of T_1 and T_2 in L_1 have first coordinates

$$\frac{\alpha n_2}{(m_3 - m_1)n_2 - (m_2 - m_1)n_3} \quad \text{and} \quad \frac{\beta n_2}{(m_4 - m_1)n_2 - (m_2 - m_1)n_4}$$

respectively, thus since $v \in \mathsf{L}_1$ is a common vertex to each of T_1 and T_2, the latter first coordinates are equal. We deduce from $0 < \alpha < \beta$ that

$$m_4n_2 - m_2n_4 - (m_3n_2 - m_2n_3) > m_1(n_3 - n_4).$$

This is a contradiction to $m_1 > 0$, $n_4 < n_3$ and $m_4n_2 - m_2n_4 > m_3n_2 - m_2n_3$.
This proves Proposition 6.58 in the case where $v \in \mathsf{L}_1$.

6.6.1.2 The origin of the base fan is the only intersection point of type (III)

Similarly to the the case where $v \in \mathsf{L}_0$, we make a simple analysis on $f_{0,t}$, $f_{2,t}$ and on the reduced system of (6.6.3) with respect to v_0. This analysis is based on the inequalities between α, β, γ_0 and γ_2. The cases from **i)** to **iv)** are the same that been considered in the case where $v \in \mathsf{L}_0$. The entries appearing in the following table represent the maximum number of positive solutions of (6.6.1) with valuations in the associated cell of $T_1 \cap T_2$.

Intersection Locus	**i)**	**ii)**	**iii)**	**iv)**
$\{v_0\}$	0	1	2	3 | 2
$\mathfrak{E}_0$	3	2	2	1 | 1
$\mathfrak{E}_2$	2	2	1	1 | 2

Assume furthermore that (6.6.1) has the maximal number five of positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\}$. Then Γ_0 and Γ_2 are both optimally sloped, and thus, since $\alpha < \beta$, we have

$$n_4 < n_3 \quad \text{and} \quad m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3. \tag{6.6.8}$$

These assumptions give the two following results.

Lemma 6.59. *The tropical curve T_1 has a vertex on L_1 iff T_2 has a vertex on L_1.*

Proof. We argue by contradiction. Assume first that T_2 has a vertex v_2 in L_1 and T_1 has no vertex in the same 1-cone. Then the points (m_3, n_3) and (m_4, n_4) are situated on different sides of the line L containing the points $(0, m_1)$ and (m_2, n_2) as shown in Figure 6.27.

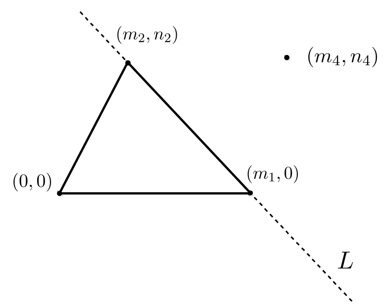

Figure 6.27: The point (m_4, n_4) is not on the same side of L as (m_3, n_3)

This disposition gives the inequalities

$$(m_2 - m_1)n_3 - (m_3 - m_1)n_2 > 0 \quad \text{and} \quad (m_2 - m_1)n_4 - (m_4 - m_1)n_2 < 0,$$

and thus we get $m_3n_2 - m_2n_3 - (m_4n_2 - m_2n_4) < m_1(n_4 - n_3)$. Moreover, since $m_1 > 0$ and $n_4 < n_3$, we get $m_3n_2 - m_2n_3 < m_4n_2 - m_2n_4$, a contradiction to (6.6.8).

Assume now that T_1 has a vertex v_1 in L_1 and T_2 has no vertex in the same 1-cone. The disposition of (m_3, n_3) and (m_4, n_4) with respect to L is the opposite of that represented in Figure 6.27. Therefore, the point (m_3, n_3) belongs to $C \cup D \cup E$ represented in Figure 6.19 (the point (m_3, n_3) cannot be situated in G since otherwise T_1 would not have a vertex $v_1 \neq v_0$). Moreover, the only way to have a transversal intersection in C_1 and C_2 is for T_2 to have a vertex on L_0 and L_1, thus (m_4, n_4) belongs to the region A of Figure 6.19. It turns out that if (m_3, n_3) belongs to any of the three regions C, D and E, it cannot produce a transversal intersection point in C_1 and C_2 simultaneously (see Figure 6.28).

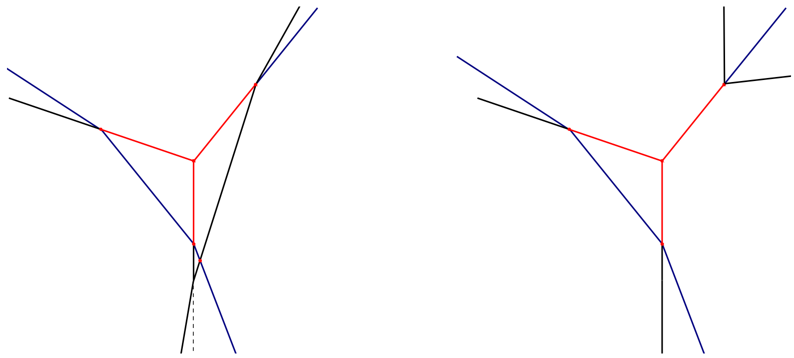

Figure 6.28: The left side represents $T_1 \cup T_2$ when $(m_3, n_3) \in E$ and the right side represents $T_1 \cup T_2$ when $(m_3, n_3) \in D$.

□

Lemma 6.60. *If T_1 has a vertex $v_1 \in \mathring{\mathsf{L}}_1$ and T_2 has a vertex $v_2 \in \mathring{\mathsf{L}}_1$, then the first coordinate of v_1 is smaller than that of v_2.*

Proof. Assume that the first coordinate of the vertex $v_1 \in \mathring{\mathsf{L}}_1$ of T_1 is greater than that of $v_2 \in \mathring{\mathsf{L}}_2$ of T_2, we prove that this gives a contradiction. Then these first coordinates satisfy

$$\frac{\alpha n_2}{n_2(m_3 - m_1) - n_3(m_2 - m_1)} > \frac{\beta n_2}{n_2(m_4 - m_1) - n_4(m_2 - m_1)} > 0.$$

Since $0 < \alpha < \beta$ and $m_1 > 0$, we have

$$n_2(m_4 - m_1) - n_4(m_2 - m_1) > n_2(m_3 - m_1) - n_3(m_2 - m_1) > 0.$$

The latter inequality induces $m_4n_2 - m_2n_4 > m_3n_2 - m_2n_3$, a contradiction to (6.6.8). □

Recall that by assumption, the system (6.6.1) has five positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\}$ and prove that this gives a contradiction. Assume furthermore that the curves T_1 and T_2 intersect transversally at $p_1 \in \mathsf{C}_1$ and $p_2 \in \mathsf{C}_2$. We consider two cases.

• **First case:** Assume that T_1 has a vertex $v_1 \in \mathsf{L}_1$. Then by Lemma 6.59, the tropical curve T_2 has a vertex v_2 in L_1, and thus by Lemma 6.60, the first coordinate of v_1 is smaller than that of v_2. Therefore, the transversal intersections $p_1 \in \mathring{\mathsf{C}}_1$ and $p_2 \in \mathring{\mathsf{C}}_2$ exist only if the point (m_3, n_3) is contained inside the triangle $(m_1, 0)$, (m_2, n_2) and (m_4, n_4) (see Figure 6.29). Such a restriction gives the inequalities

$$(m_3 - m_1)n_4 - (m_4 - m_1)n_3 < 0 \quad \text{and} \quad (m_3 - m_2)(n_4 - n_2) - (m_4 - m_2)(n_3 - n_2) > 0,$$

, from which we deduce $m_4n_2 - m_2n_4 - m_3n_2 + m_2n_3 > m_1(n_3 - n_4)$. A contradiction to (6.6.8).

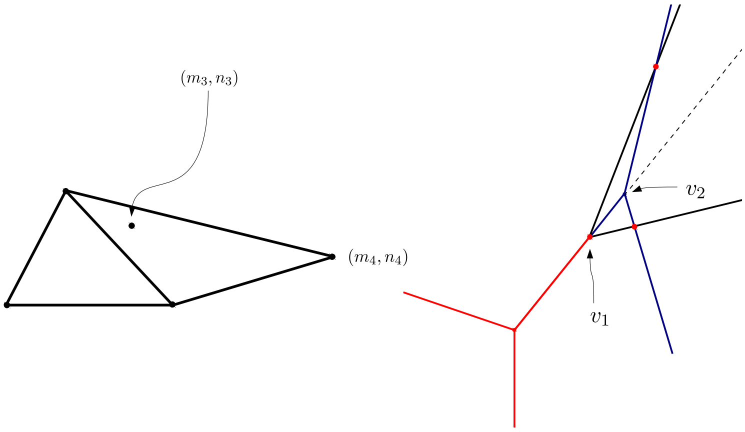

Figure 6.29: Location of (m_3, n_3) in order for T_1 and T_2 to have two transversal intersection points.

• **Second case:** Assume now that T_1 does not have a vertex in L_1. Then Lemma 6.59 shows that T_2 does not have a vertex in L_1. Note that since $p_1 \in \mathsf{C}_1$ and $p_2 \in \mathsf{C}_2$, each of T_1 and T_2 has one edge in each of these 2-cones, and thus both (m_3, n_3) and (m_4, n_4) belong to the region A represented in Figure 6.19. Therefore, we have the following inequalities

$$m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0 \quad \text{and} \quad n_4 < n_3 < 0.$$

In what follows in this subsection, we make a case-by-case study on the reduced system with respect to v_0. We prove in each one of the following cases that (6.6.1) cannot have *five* non-degenerate positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\}$, and *two* non-degenerate positive solutions, each with valuation in p_1 and p_2. Recall that by assumption, each of Γ_0 and Γ_2 are both optimally sloped.

i) Assume that there exists only one element of the set $\{\alpha, \gamma_0, \gamma_2\}$ that is equal to $\min(\alpha, \gamma_0, \gamma_2)$. Recall that the reduced system of (6.6.3) with respect to v_0 has *no* real positive solutions. Since Γ_0 is optimally sloped, we have $\gamma_2 < \gamma_0 < \alpha < \beta$ (Γ_0 in this case looks similar to what is represented in Figure 6.20, where the only difference is that the dotted line does not intersect the origin of the axis). Recall that $n_4 < n_3 < 0 < n_2$ and by assumption both $\text{coef}(a_3)$ and $\text{coef}(b_4)$ are negative, thus by Descartes' rule of sign applied to $f_{0,t}$, we have $\text{coef}(c_2) > 0$. Therefore, using the same rule on $f_{2,t}$, we deduce that the latter polynomial has at most *one* positive solution. Indeed, since $m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0$ and $c = \text{coef}(c_2) > 0$. We conclude that (6.6.1) has at most *one* positive solution with valuation in $\mathring{\mathfrak{E}}_2$, and thus the latter system has at most *four* positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v_0\}$, a contradiction.

ii) Assume that $\gamma_0 = \gamma_2 < \alpha$. Recall that the reduced system of (6.6.3) (see the system (6.4.15) in Subsection 6.4.3) with respect to v_0 has at most *one* positive solution. Since Γ_0 contains an horizontal edge, each of $\mathring{\mathfrak{C}}_0$ and $\mathring{\mathfrak{C}}_2$ contains at most *two* positive solutions. If $\operatorname{coef}(c_0) = -\operatorname{coef}(c_2)$, then the reduced system of (6.6.3) has *no* positive solutions and $\mathring{\mathfrak{C}}_0 \cup \mathring{\mathfrak{C}}_2 \cup \{v_0\}$ has the valuations of at most *four*, and we are done. Assume that the reduced system of (6.6.3) has *one* positive solution, then $\operatorname{coef}(c_0)\operatorname{coef}(c_2) < 0$. Moreover, if $\mathring{\mathfrak{C}}_0$ (resp. $\mathring{\mathfrak{C}}_2$) contains the valuations of *two* positive solutions, then in order for the two binomials of $f_{0,t}$ (resp. $f_{2,t}$), associated to the edges with negative slope of Γ_0 (resp. Γ_2), to have non-degenerate positive solutions, we have $\operatorname{coef}(c_0) < 0$ (resp. $c = \operatorname{coef}(c_2) - \operatorname{coef}(c_0) < 0$). Indeed, since $m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0$ and $n_4 < n_3 < 0$. Therefore $\operatorname{coef}(c_2) < \operatorname{coef}(c_0) < 0$, a contradiction to $\operatorname{coef}(c_0)\operatorname{coef}(c_2) < 0$.

iii) Assume that $\alpha = \gamma_0 < \beta < \gamma_2$ (for the case where $\alpha = \gamma_2 < \beta < \gamma_0$ we proceed with the same type of arguments as in **iii)** to find the same contradiction). Recall that the reduced system of (6.6.3) (see the system (6.4.16) in Subsection 6.4.3) with respect to v_0 has at most *two* positive solutions. Since $n_2 > 0$ and $\alpha = \gamma_0 < \gamma_2$, we have that Γ_0 contains only *one* edge with a negative slope (see Figure 6.23). Moreover, since $\delta = \gamma_0$ and $m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3 < 0$, then also Γ_2 contains only *one* edge with negative slope. Therefore $\mathring{\mathfrak{C}}_0 \cup \mathring{\mathfrak{C}}_2 \cup \{v_0\}$ contains the valuations of at most *four* positive solutions, a contradiction.

iv) Assume that $\alpha = \gamma_0 = \gamma_2 < \beta$. Recall that the reduced system of 6.6.3 with respect to v_0 (see (6.4.18) of Subsection 6.4.3) has at most *three* positive solutions. This system is supported on a circuit, where the point $(0,0)$ is contained in the triangle with vertices $(m_1, 0)$, (m_2, n_2) and (m_3, n_3). Indeed, since from $n_3 < 0$ and $m_3n_2 - m_2n_3$, the point (m_3, n_3) is contained in region A represented in Figure 6.19. Therefore, (6.4.18) has at most *two* positive solutions. Moreover, the relation $\alpha = \gamma_0 = \gamma_2 \leq \delta$ shows that each of Γ_0 and Γ_2 contains only *one* edge with negative slope such that the associated facial subpolynomials is a binomial (see Figure 6.30). Therefore, $\mathfrak{C}$ contains the valuation of at most *four* positive solutions of (6.6.1), a contradiction.

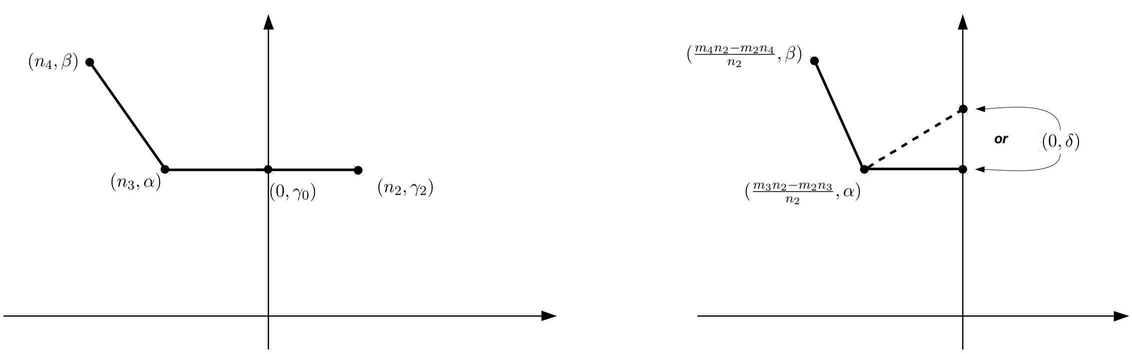

Figure 6.30: Examples of Γ_0 and Γ_2 for $\alpha = \gamma_0 = \gamma_2 < \beta$.

We conclude that Theorem 6.57 is proved for $0 < \alpha < \beta$.

6.6.2 The case $\alpha = 0 < \beta$

The tropical curve T_1 has only one vertex v_0, thus this vertex is the only non-transversal intersection point of type (III) of T_1 and T_2. Moreover, the reduced system with respect to v_0 is

$$-1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} + \mathrm{coef}(a_3) y_1^{m_3} y_2^{n_3} = -1 + y_1^{m_1} + y_1^{m_2} y_2^{n_2} = 0$$

and does *not* have non-zero solutions. Therefore, the valuation of any positive solution of (6.6.1) is either a transversal intersection point of T_1 and T_2 or it is contained in a cell of type (I) that belongs to a 1-cone of $\mathcal{E}$. From $\alpha = 0$, we deduce that T_1 and T_2 intersect transversally in at most *two* points. Indeed, this comes from applying Lemma 6.22 on T_2 since T_1 has at most two edges different from any 1-cone of $\mathcal{E}$ (see Figure 6.31). Therefore, since each $f_{0,t}$ and $f_{2,t}$ has at most *three* and *two* positive solutions respectively, the system (6.6.1) cannot have more than seven positive solutions.

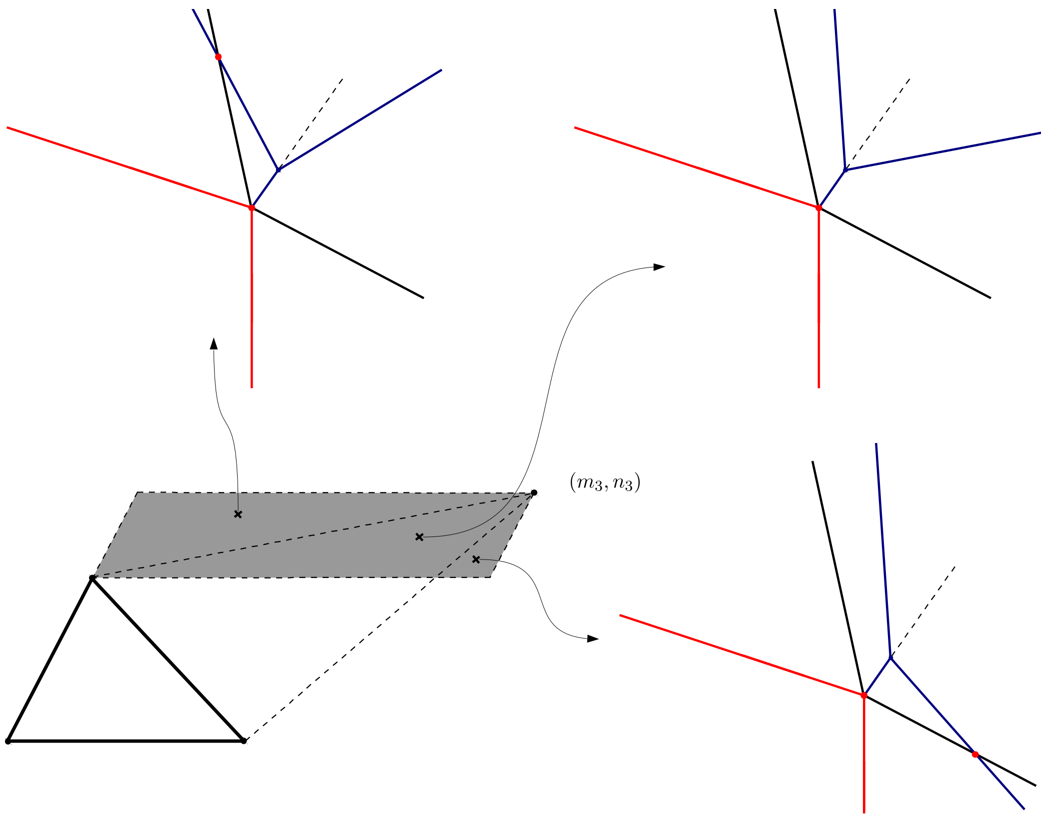

Figure 6.31: If (m_4, n_4) belongs to the grey area, then T_1 and T_2 do not intersect transversally at two points.

Assume that the latter system has seven positive solutions. We show that this gives a contradiction. Then T_1 and T_2 intersect transversally at *two* points and $\mathring{\mathfrak{C}}_0$ (resp. $\mathring{\mathfrak{C}}_2$) contains the valuations of *three* (resp. *two*) non-degenerate positive solutions of (6.6.1). This shows that Γ_0 and Γ_2 are both optimally sloped, and thus, since $\alpha < \min(\gamma_0, \gamma_2, \beta)$, we have $n_3 > \max(n_2, n_4)$ and $m_3n_2 - m_2n_3 > \max(0, m_4n_2 - m_2n_4)$. Therefore, the point (m_3, n_3) belongs to the region $D_{1,1}$ represented in Figure 6.37 (see page 143). This gives that the tropical curve T_1 has one edge belonging to each of $\mathring{\mathsf{C}}_1$ and $\mathring{\mathsf{C}}_2$ (see Figure 6.31). Hence, Proposition 6.27 implies that, since $\mathrm{coef}(a_2) = 1$ and T_1 intersects T_2 at two transversal points which are valuations of positive solutions of (6.6.1), we have $\mathrm{coef}(a_3) < 0$ and $\mathrm{coef}(b_4) < 0$. Therefore, Descartes' rule of sign applied to $f_{2,t}$, which has three positive solutions, shows that $0 < m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3$. Then we get $\delta > \beta > \alpha$, and from $\gamma_0 > \gamma_2 > \beta > \alpha$, we deduce that $n_2 < n_4 < n_3$. Fixing (m_3, n_3) in the region $D_{1,1}$ represented in Figure 6.37, we deduce that (m_4, n_4) belongs to the grey region shown in Figure 6.31. Moreover, since the first coordinate of the vertex $v_2 \in \mathsf{L}_1$ of T_2 is positive (see Figure 6.31), the curves T_1 and T_2 intersect transversally in at most *one* point, a contradiction.

6.6.3 The case $\alpha < 0 < \beta$.

Since $\alpha < 0$, the tropical curve T_1 does not have a vertex at the origin v_0 of $\mathcal{E}$, and thus there does not exist a non-transversal tropical intersection point in this origin.

Assume first that T_1 and T_2 intersect non-transversally at a cell $\mathfrak{E}_0$ of type (I) in L_0 and that the latter curves do not intersect non-transversally in a cell of type (I) in L_2. If there exists a non-transversal intersection point v contained in any 1-cone of $\mathcal{E}$, then Theorem 6.57 is proved for $\alpha < 0 < \beta$. Indeed, Remark 6.42 of Subsection 6.4.4 shows that the reduced system with respect to v has at most *two* positive solutions. Moreover, Lemma 6.44 from the same Subsection shows that there exists at most *one* transversal intersection p. Therefore, the system (6.6.1) has at most *one* (resp. *two*, *three*) positive solutions with valuation in p (resp. v, $\mathring{\mathfrak{E}}_0$), and we are done. Theorem 6.57 comes as a consequence of Theorem 6.15 also in the case where there does not exist such v.

In what follows in this Subsection we assume that T_1 and T_2 intersect non-transversally in *two* cells $\mathfrak{E}_0 \subset \mathsf{L}_0$ and $\mathfrak{E}_2 \subset \mathsf{L}_2$ of type (I).

6.6.3.1 There exists a non-transversal intersection of type (III)

Then this non-transversal intersection point v of type (III) is contained in the 1-cone L_1. Indeed, since otherwise one of $\mathfrak{E}_0$ or $\mathfrak{E}_2$ would not exist (see Figure 6.32 for example).

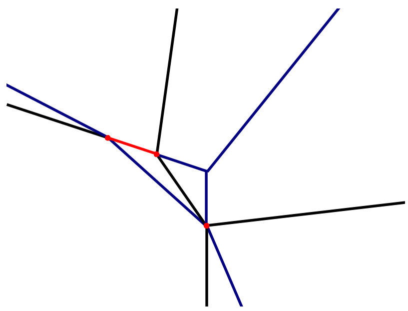

Figure 6.32: When $\alpha < 0 < \beta$, if $v \in \mathsf{L}_0$, then there does not exist a cell of type (I) in L_0.

Since $v \in \mathsf{L}_1$ is the common vertex of T_1 and T_2 that has a positive first coordinate, and

$\alpha < 0 < \beta$, we deduce

$$n_2(m_3 - m_1) - n_3(m_2 - m_1) < 0 \quad \text{and} \quad n_2(m_4 - m_1) - n_4(m_2 - m_1) > 0.$$

Computing the difference we get

$$m_4n_2 - m_2n_4 - (m_3n_2 - m_2n_3) > m_1(n_3 - n_4), \tag{6.6.9}$$

and thus, since $m_1 > 0$, we have $n_4 < n_3 \Rightarrow m_3n_2 - m_2n_3 < m_4n_2 - m_2n_4$. Moreover, since $\alpha < 0 < \beta$, if $n_3 < n_4$ (resp. $m_3n_2 - m_2n_3 < m_4n_2 - m_2n_4$), then $\mathring{\mathfrak{C}}_0$ (resp. $\mathring{\mathfrak{C}}_2$) contains the valuations of at most *two* (resp. *one*) positive solution. Indeed, the lower hull Γ_0 (resp. Γ_2) has at least one edge with non-negative slope (see Figure 6.33 for an example), and thus is not optimally sloped.

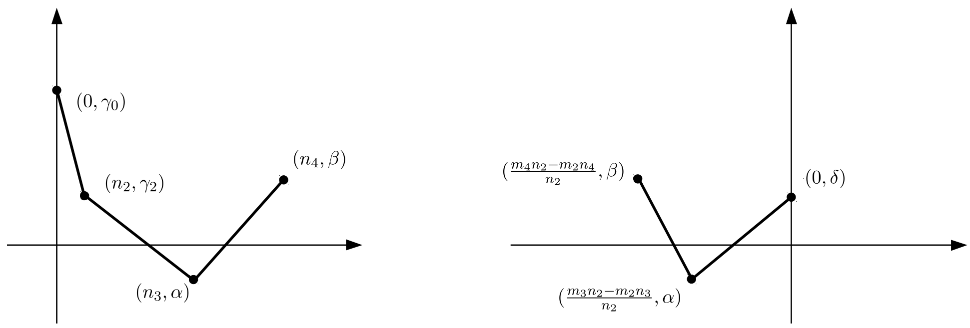

Figure 6.33: Examples of Γ_0 and Γ_2 not being optimally sloped for $\alpha < 0 < \beta$.

Therefore, $\mathring{\mathfrak{C}}_0 \cup \mathring{\mathfrak{C}}_2$ cannot contain the valuations of more than *four* positive solutions. Lemma 6.44 shows that there can exist at most *one* positive transversal intersection that can be contained only in $\mathring{\mathsf{C}}_0$.

Assume first that T_1 and T_2 intersect transversally at a point $p \in \mathring{\mathsf{C}}_0$ and (6.6.1) has a positive solution with valuation p. Then, since $\text{coef}(a_0) = -1$, Proposition 6.27 shows that both $\text{coef}(a_3)$ and $\text{coef}(b_4)$ are positive. Therefore, the system

$$y_1^{m_1} + y_1^{m_2}y_2^{n_2} + \text{coef}(a_3)y_1^{m_3}y_2^{n_3} = y_1^{m_1} + y_1^{m_2}y_2^{n_2} + \text{coef}(b_4)y_1^{m_4}y_2^{n_4} = 0. \tag{6.6.10}$$

does *not* have positive solutions, and thus (6.6.1) has at most *five* positive solutions.

Assume now that both $\text{coef}(a_3)$ and $\text{coef}(b_4)$ are negative. Then by Proposition 6.27, the system (6.6.1) does *not* have a positive solution with valuation in $\mathring{\mathsf{C}}_0$. Moreover, the system (6.6.10) has at most *two* positive solutions. Therefore, the system (6.6.1) has at most *six* positive solutions.

6.6.3.2 There does not exist an intersection point of type (III)

Recall that by assumption, we have $\mathfrak{C}_0 \subset \mathsf{L}_0$ and $\mathfrak{C}_2 \subset \mathsf{L}_2$. This means that, since $\alpha < 0$, the tropical curve T_1 has one vertex in L_0 and one vertex in L_2. Therefore the point (m_3, n_3) is contained in the region $D \cup G$ represented in Figure 6.19, where $n_3 > 0$ and $m_3n_2 - m_2n_3 > 0$. If (6.6.1) has more than *six* positive solutions in total, then T_1 and T_2 intersect transversally in at least *two* points. Indeed, since the only other solutions have valuations contained in $\mathring{\mathfrak{C}}_0 \cup \mathring{\mathfrak{C}}_2$, where the latter contains the valuations of at most *five* non-degenerate positive solutions of (6.6.1).

We prove Theorem 6.57 by contradiction. Assume that (6.6.1) has *five* positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$ and *two* positive ones with valuations transversal intersections p_1 and p_2. Recall that $\mathrm{coef}(a_0) = -1$ and $\mathrm{coef}(a_2) = 1$. Then from Proposition 6.27, we have $p_1 \in \mathsf{C}_1$ and $p_2 \in \mathsf{C}_2$, so that both $\mathrm{coef}(a_3)$ and $\mathrm{coef}(b_4)$ are negative. Since $\mathring{\mathfrak{E}}_2$ contains the valuations of *two* non-degenerate positive solutions of (6.6.1) (by assumption), both edges of Γ_2 have negative slopes. Moreover, Descartes' rule of sign applied to $f_{2,t}$ shows that since $\mathrm{coef}(a_3)\,\mathrm{coef}(b_4) > 0$, we have $0 < m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3$ and thus $\delta > \beta$ (see Figure 6.34). Note that, since $\mathring{\mathfrak{E}}_0$ contains the valuations of *three* positive solutions of (6.6.1), all the edges of Γ_0 have negative slopes, and thus $\gamma_0 > \gamma_2$ (recall that $n_2 > 0$). From $\alpha < 0 < \beta < \gamma_2 < \gamma_0$, we deduce that $0 < n_2 < n_4 < n_3$.

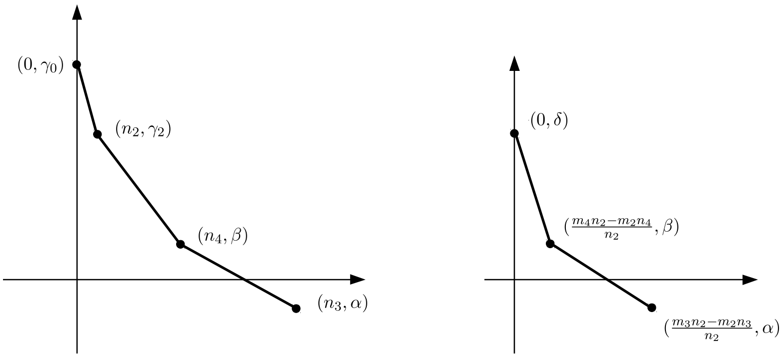

Figure 6.34: Examples of optimally sloped Γ_0 and Γ_2 for $\alpha < 0 < \beta$.

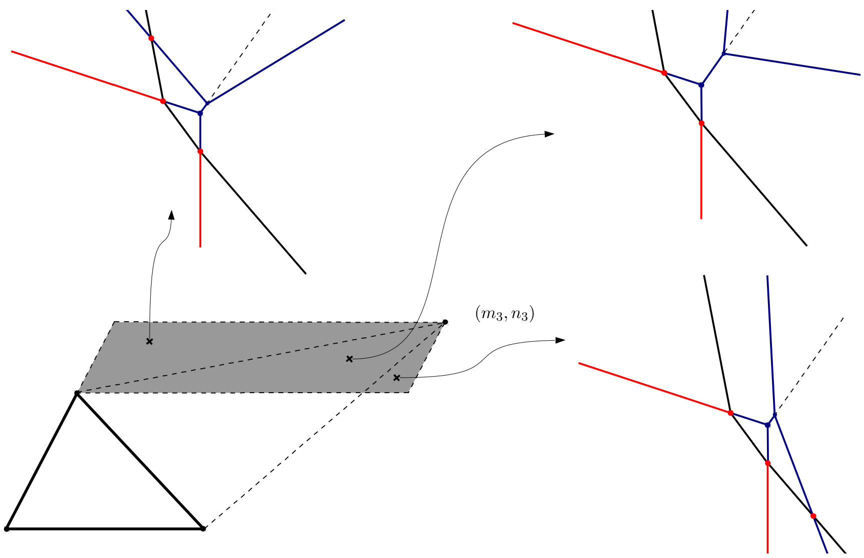

Figure 6.35: If (m_4, n_4) belongs to the grey region, then T_1 does not intersect T_2 at two transversal intersection points.

We deduce that the points (m_3, n_3) and (m_4, n_4) are contained in the region $D_{1,1}$ represented in Figure 6.37 (see page 143). Indeed, since for $i = 3, 4$, we have $0 < n_2 < n_i$ and $m_in_2 - m_2n_i >$

$0 > m_1(n_2 - n_i)$. Moreover, since $\alpha < 0$ and $\beta > 0$, the tropical curve T_1 does not have a vertex in L_1, and the only vertices of T_2 are v_0 (the origin of $\mathcal{E}$) and $v_2 \in \mathsf{L}_1$. Recall that $n_4 < n_3$ and $m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3$, thus fixing (m_3, n_3) in the region $D_{1,1}$ represented in Figure 6.37, the point (m_4, n_4) belongs to the grey region appearing in Figure 6.35. We deduce that if (m_4, n_4) belongs to the grey region, the curves T_1 and T_2 do not intersect transversally in each of C_1 and C_2, a contradiction.

6.6.4 The case $\alpha < 0$ and $\beta = 0$

The tropical curve T_2 has only one vertex in the origin of $\mathcal{E}$, thus T_1 and T_2 do not intersect non-transversally in points of type (III). We prove Theorem 6.57 by contradiction. Similarly to the case $\alpha = 0$ and $\beta > 0$, we assume that (6.6.1) has *seven* positive solutions such that *two* of them have valuations which are transversal intersections and $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$) contains the valuations of *three* (resp. *two*) non-degenerate positive solutions. Since each of Γ_0 and Γ_2 are optimally sloped, we have $n_2 < n_4 < n_3$, $\gamma_0 > \gamma_2 > \beta > \alpha$ and $0 < m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3$. Theorem 6.57 then is proved by applying similar arguments used in the case where $\alpha = 0$ and $\beta > 0$.

6.6.5 The case $\alpha < \beta < 0$.

Using the same arguments as in the case $\alpha < 0 < \beta$, we assume in what follows in this subsection that T_1 and T_2 intersect non-transversally at cells $\mathfrak{E}_0 \in \mathsf{L}_0$ and $\mathfrak{E}_2 \in \mathsf{L}_2$ of type (I). Since $\mathcal{E}$ is a base fan of T_1 (resp. T_2) and $\alpha < 0$ (resp. $\beta < 0$), the latter assumption means that T_1 (resp. T_2) has a vertex on each of L_0 and L_2. Therefore, we have $0 < \min(n_3, n_4)$ and $0 < \min(m_3n_2 - m_2n_3, m_4n_2 - m_2n_4)$.

6.6.5.1 There exists a non-transversal intersection point of type (III)

We distinguish two cases for a non-transversal intersection point v of type (III).

• **First case: $v \in \mathsf{L}_1$.** Then, both (m_3, n_3) and (m_4, n_4) are contained in the region G represented in Figure 6.19. Indeed, since both α and β are negative and $\mathfrak{E}_0 \subset \mathring{\mathsf{L}}_0$, $\mathfrak{E}_2 \subset \mathring{\mathsf{L}}_2$, $v \in \mathring{\mathsf{L}}_1$, each of T_1 and T_2 has a vertex in the relative interior of each 1-cone of $\mathcal{E}$.

Theorem 6.57 becomes trivial if $\mathrm{coef}(a_3)$ or $\mathrm{coef}(b_4)$ is positive. Indeed, otherwise the reduced system (6.6.10) with respect to v would *not* have positive solutions. Moreover, by Lemma 6.44, the curves T_1 and T_2 intersect transversally in at most *one* point. Therefore, since $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$) contains the valuations of at most *three* (resp. *two*) positive solutions, the total number of positive solutions of (6.6.1) is at most *six*.

Assume that both $\mathrm{coef}(a_3)$ and $\mathrm{coef}(b_4)$ are negative. In what follows, we assume that (6.6.1) has more than *six* positive solutions and prove that this gives a contradiction. Lemma 6.44 shows that if T_1 and T_2 intersect transversally in a point p_0 (which is the maximal number of such intersection points), then p_0 is contained in C_0. However Proposition 6.27 shows that since $\mathrm{coef}(a_3) < 0$, $\mathrm{coef}(b_4) < 0$, $\mathrm{coef}(a_0) = -1$ and $\mathrm{coef}(b_0) = -1$, this point p_0 is not the valuation of a positive solution of (6.6.1). Therefore, the only possible way for (6.6.1) to have more than six non-degenerate positive solutions, is for it to have *seven* non-degenerate positive solutions satisfying that $\mathring{\mathfrak{E}}_0$ (resp. $\mathring{\mathfrak{E}}_2$, $\{v\}$) contains the valuation of three (resp. two, two) positive solutions. This shows that Γ_0 and Γ_2 are both optimally sloped, and since $\alpha < \beta < 0$, we have $0 < n_2 < n_4 < n_3$ and $0 < m_4n_2 - m_2n_4 < m_3n_2 - m_2n_3$. However this contradicts the fact that both of (m_3, n_3) and

(m_4, n_4) are contained in the region G represented in Figure 6.19, and we are done.

• **Second case:** $\boldsymbol{v \in \mathring{\mathsf{L}}_0}$. We have $n_4\alpha = n_3\beta$ (a vertex of T_1 coincides with a vertex of T_2, both in $\mathring{\mathsf{L}}_0$), and since $\alpha < \beta < 0$ and $0 < \min(n_3, n_4)$, we get $0 < n_4 < n_3$. Moreover, since both γ_0 and γ_2 are positive, from Remark 6.36, the lower hull Γ_0 contains an edge e_1 adjacent to the points (n_4, β), (n_3, α), and with negative slope (c.f. Figure 6.36).

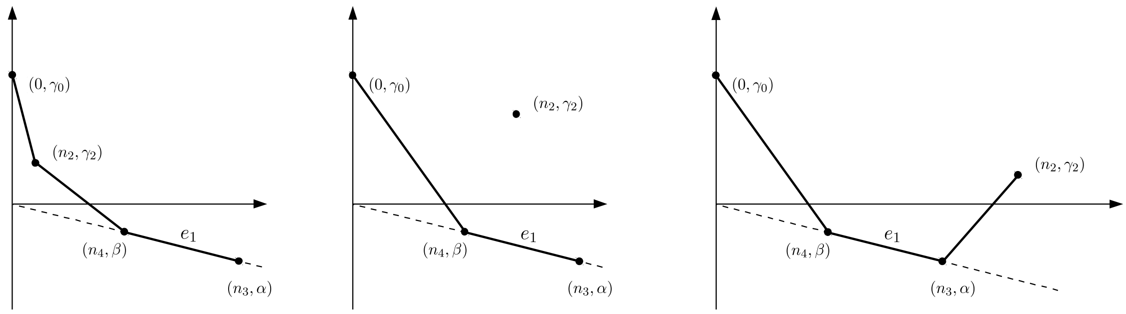

Figure 6.36: Examples of Γ_0 for $\alpha < \beta < 0$.

The facial subpolynomial $f_0^{(1)}(y) = -\operatorname{coef}(a_3)y^{n_3} + \operatorname{coef}(b_4)y^{n_4}$ (which is associated to e_1) is obtained from $f_{0,t}(t^{-\lambda_1}y)/t^{\mu_1}$, where $\lambda_1 = \beta/n_4$ and $\mu_1 = 0$. Therefore, by Corollary 6.12 of Section 6.2, if y_0 is a largely ordered positive root of $f_{0,t}$, then $\operatorname{coef}(y_0)$ is not a positive root of $f_0^{(1)}$. This shows that $\mathring{\mathfrak{E}}_0$ contains the valuations of at most *two* positive solutions of (6.6.1). Moreover, if the latter system has *two* positive solutions with valuations in $\mathring{\mathfrak{E}}_0$, then $0 < n_2 < n_4$. Indeed, otherwise the point (n_2, γ_2) is not a vertex of Γ_0, or Γ_0 has an edge with positive slope (c.f. Figure 6.36 the center and right).

Recall that since $v \in \mathring{\mathsf{L}}_0$, if the reduced system with respect to v has positive solutions, then their number is equal to that of the positive ones of

$$-1 + y^{m_1} + d_3 y^{\frac{m_3 n_4 - m_4 n_3}{n_4 - n_3}} = 0. \tag{6.6.11}$$

Note that by Lemma 6.44, the curves T_1 and T_2 intersect transversally in at most *one* point, and if such intersection point exists, it is contained in $\mathring{\mathsf{C}}_2$. We assume that (6.6.1) has more than *six* positive solutions and prove that this gives a contradiction. Then (6.6.1) has *six* positive solutions with valuations in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2 \cup \{v\}$ (which is the maximum number) and *one* positive solution with valuation a transversal intersection $p \in \mathring{\mathsf{C}}_2$. We deduce from the latter condition and Proposition 6.27 that both $\operatorname{coef}(a_3)$ and $\operatorname{coef}(b_4)$ are negative. Moreover, since $\alpha < \beta < 0$ and each of $\mathring{\mathfrak{E}}_0$ and $\mathring{\mathfrak{E}}_2$ contains the valuations of *two* positive solutions (which is the maximum), we deduce $0 < n_2 < n_4 < n_3$ and $0 < m_4 n_2 - m_2 n_4 < m_3 n_2 - m_2 n_3$. Since d_3 has the same sign as $\operatorname{coef}(a_3)$ (c.f. (6.4.25)), and (6.6.11) has the maximal number *two* of positive solutions, by Descartes' rule of signs we have $(m_3 n_4 - m_4 n_3)/(n_4 - n_3) > 0$, which together with $0 < n_4 < n_3$ implies that

$$m_3 n_4 - m_4 n_3 < m_1(n_4 - n_3) < 0. \tag{6.6.12}$$

Therefore, the points (m_3, n_3) and (m_4, n_4) are contained in the region $D_{1,1}$ represented in Figure 6.37, thus fixing (m_3, n_3) in the region $D_{1,1}$, we deduce that (m_4, n_4) belongs to the region $D_{1,2}$ represented in Figure 6.37.

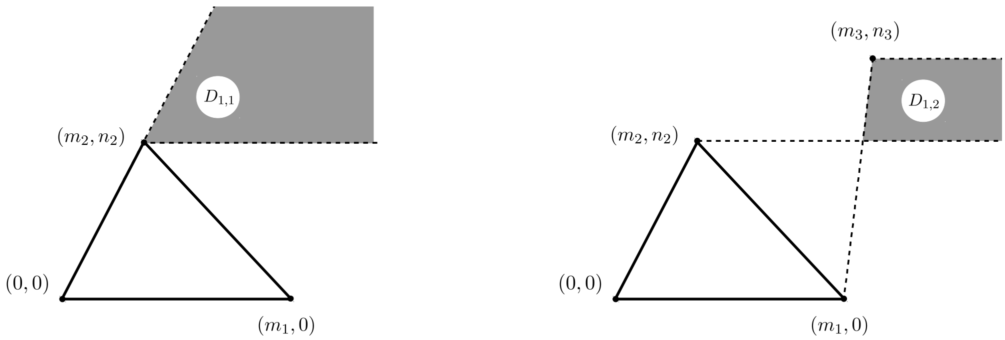

Figure 6.37: On the left: Region $D_{1,1}$, and on the right: Region $D_{1,2}$.

Therefore, if there is a transversal intersection point in C_2, then the first coordinate of the vertex $v_1 \in \mathsf{L}_2$ of T_1 is bigger than that of the vertex $v_2 \in \mathsf{L}_2$ of T_2 (See Figure 6.38).

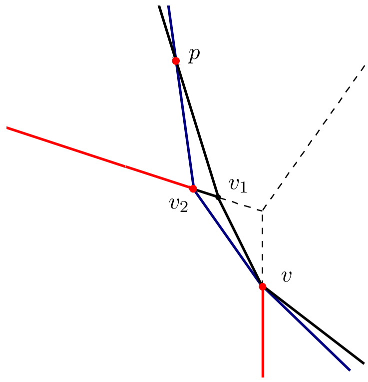

Figure 6.38: The curves T_1 and T_2 for $0 < \alpha < \beta$.

This means that $\frac{\beta}{m_4 n_2 - m_2 n_4} < \frac{\alpha}{m_3 n_2 - m_2 n_3}$. Finally, recall that $\alpha n_4 = \beta n_3$, therefore we get $m_3 n_4 > m_4 n_3$, a contradiction to (6.6.12).

6.6.5.2 There does not exist an intersection point of type (III)

Assume that (6.6.1) has more than *six* positive solutions, we prove that this leads to a contradiction. Then it has *five* positive solutions with valuation in $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$ (which is the maximal number) and T_1, T_2 intersect transversally in *two* points p_1 and p_2 so that each one is the valuation of a positive solution. Since $\text{coef}(a_0) = \text{coef}(b_0) < 0$ and $\text{coef}(a_2) = \text{coef}(b_2) > 0$, Proposition 6.27 shows that $p_1 \in \mathsf{C}_1$ and $p_2 \in \mathsf{C}_2$.

Recall that $0 < \min(n_3, n_4)$ and $0 < \min(m_3 n_2 - m_2 n_3, m_4 n_2 - m_2 n_4)$. Since $\alpha < \beta < 0 < \min(\gamma_0, \gamma_2)$ and each of $\mathring{\mathfrak{E}}_0$ and $\mathring{\mathfrak{E}}_2$ contains the valuations of respectively *three* and *two* positive solutions of (6.6.1), we have

$$0 < n_2 < n_4 < n_3 \quad \text{and} \quad 0 < m_4 n_2 - m_2 n_4 < m_3 n_2 - m_2 n_3. \tag{6.6.13}$$

Indeed, since each of Γ_0 and Γ_2 are optimally sloped (see Figure 6.39 for an example).

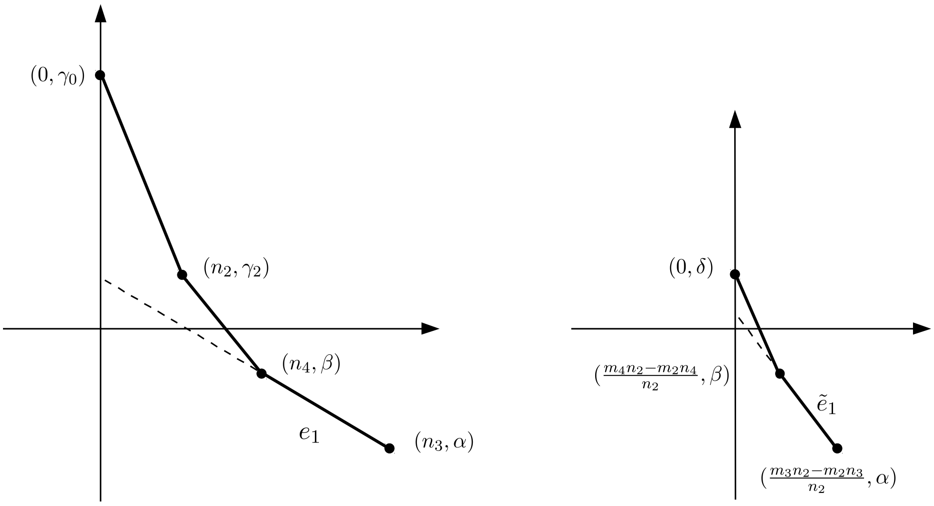

Figure 6.39: The graphs Γ_0 and Γ_2, having three edges with negative slope for $0 < \alpha < \beta$.

Therefore, from Remark 6.36 of Subsection 6.4.1, the lower hull Γ_2 (resp. Γ_0) has an edge $\tilde{e}_1$ (resp. e_1) with negative slope $n_2(\alpha - \beta)/\left((m_3 - m_4)n_2 - (n_3 - n_4)m_2\right)$ (resp. $(\alpha - \beta)/(n_3 - n_4)$). The facial subpolynomial $f_2^{(1)}(y)$ (resp. $f_0^{(1)}(y)$), which is associated to $\tilde{e}_1$ (resp. e_1), is obtained from $f_{2,t}(t^{-\tilde{\lambda}_1} y)/t^{\tilde{\mu}_1}$ (resp. $f_{0,t}(t^{-\lambda_1} y)/t^{\mu_1}$), where

$$\tilde{\lambda}_1 = \frac{(\alpha - \beta)n_2}{(m_3 - m_4)n_2 - (n_3 - n_4)m_2} \quad \text{and} \quad \tilde{\mu}_1 = \frac{(m_3 n_2 - m_2 n_3)\alpha - (m_4 n_2 - m_2 n_4)\beta}{(m_3 - m_4)n_2 - (n_3 - n_4)m_2}$$

(resp. $\lambda_1 = (\alpha - \beta)/(n_3 - n_4)$ and $\mu_1 = (n_3\beta - n_4\alpha)/(n_3 - n_4)$). Moreover, since all roots of $f_{0,t}$ and $f_{0,t}$ are largely ordered, we have that both μ_1 and $\tilde{\mu}_1$ are positive. From $\alpha < \beta < 0$, $\mu_1, m\tilde{u}_1 > 0$ and the inequalities in (6.6.13), we deduce the inequalities

$$\frac{\alpha}{n_3} < \frac{\beta}{n_4} \quad \text{and} \quad \frac{\alpha}{m_3 n_2 - m_2 n_3} < \frac{\beta}{m_4 n_2 - m_2 n_4}. \tag{6.6.14}$$

Also from the inequalities appearing in (6.6.13), the curve T_1 (resp. T_2) has two vertices $v_1 \in \mathring{L}_0$ and $\tilde{v}_1 \in \mathring{L}_2$ (resp. $v_2 \in \mathring{L}_0$ and $\tilde{v}_2 \in \mathring{L}_2$). Therefore, from the inequalities of (6.6.14), the second coordinate of v_1 is smaller than that of the vertex v_2 and the first coordinate of the vertex $\tilde{v}_1$ is smaller than that of the vertex $\tilde{v}_2$ (see Figure 6.40).

Moreover, from inequalities appearing in (6.6.13), we deduce that the point (m_3, n_3) belongs to the region $D_{1,1}$ of Figure 6.37, and that fixing the latter point in the region $D_{1,1}$, the point (m_4, n_4) belongs to the grey region represented in Figure 6.40. However with (m_4, n_4) anywhere in the latter region, T_1 and T_2 do not intersect transversally at more than one point (see right side of Figure 6.40 for an example).

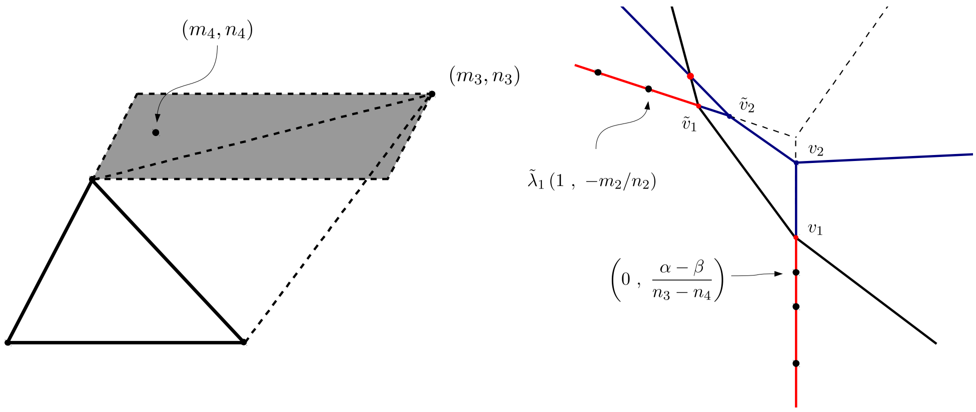

Figure 6.40: If (m_4, n_4) belongs to the grey region, then the curves T_1 and T_2 intersect in at most one transversal point.

6.6.6 The case $\alpha = \beta < 0$.

The lower hulls Γ_0 and Γ_2 have one horizontal edge each, and thus $\mathring{\mathfrak{E}}_0 \cup \mathring{\mathfrak{E}}_2$ contains the valuations of at most three positive solutions. Therefore, applying the same arguments as in the case where $\alpha < \beta < 0$, we deduce Theorem 6.57.

6.7 Proof of Theorem 6.3 (part 2).

Consider the highly non-degenerate normalized system

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0. \end{aligned} \tag{6.7.1}$$

In this Section, we prove the following result.

Theorem 6.61. *If* $\alpha\beta \neq 0$, $\operatorname{coef}(a_0)/\operatorname{coef}(b_0) \neq \operatorname{coef}(a_2)/\operatorname{coef}(b_2)$ *and* $\operatorname{coef}(a_i) \neq \operatorname{coef}(b_i)$ *for* $i = 0, 2$, *then* (6.6.1) *cannot have more than six positive solutions.*

Since $\operatorname{coef}(a_i) \neq \operatorname{coef}(b_i)$ for $i = 1, 2$, no positive solution of (6.7.1) can have valuation in a non-transversal cell of type (I). Indeed, if T_1 and T_2 intersect non-transversally at a cell $\mathfrak{E}_0$ of type (I) contained in, say L_0, then the reduced system with respect to $\mathfrak{E}_0$ is $\operatorname{coef}(a_0) + y_1^{m_1} = \operatorname{coef}(b_0) + y_1^{m_1} = 0$, which does not have any solutions. Therefore, the valuation of each positive solution is contained in one of the following.

- Non-transversal intersection point of type (III), which can either be v_0 or $v \in \mathsf{L}_i$ for some $i \in \{0, 1, 2\}$.
- Non-transversal intersection point of type (II).
- Transversal intersection point.

In what follows, we assume the hypotheses of Theorem 6.61.

6.7.1 The case $0 < \alpha \leq \beta$

Recall that there exists a non-transversal intersection point v_0 of type (III), which is the origin of $\mathcal{E}$. From Subsection 6.4.3, the inequalities on $\text{coef}(a_i)$ and $\text{coef}(b_i)$ for $i = 0, 2$ show that the reduced system of (6.7.1) with respect to v_0 has at most *one* positive solution. To prove Theorem 6.61 when $0 < \alpha < \beta$, we distinguish two cases.

6.7.1.1 There exists a non-transversal intersection point of type (III)

Without loss of generality, we may assume that the non-transversal intersection point of type (III) $v \neq v_0$ is contained in L_0. Recall from Subsection 6.4.4 that the reduced system with respect to v is a system supported on four points, thus it has at most *three* positive solutions. Moreover, the curves T_1 and T_2 intersect in at most two points of type (II) (see Figure 6.41 for example). Recall that by Lemma 6.44, the curves T_1 and T_2 have at most *one* transversal intersection point. Therefore, the system (6.7.1) cannot have more than *seven* positive solutions, and if there exists seven positive solutions, then their valuations are distributed in the following way. *Three* positive solutions with valuation $v \in \mathring{\mathsf{L}}_0$, *one* positive solution with valuation v_0, *one* positive solution with valuation a transversal intersection point $p \in \mathsf{C}_2$ (by Lemma 6.44 since $v \in \mathring{\mathsf{L}}_0$) and *two* positive solutions where each has valuation a non-transversal intersection point of type (II) $v_1 \in \mathring{\mathsf{L}}_1$ and $v_2 \in \mathring{\mathsf{L}}_2$ respectively. However, these conditions cannot be met at the same time (c.f. Figure 6.41). Indeed, since the existence of the intersection points $v \in \mathring{\mathsf{L}}_0$, $v_1 \in \mathring{\mathsf{L}}_1$ and $v_2 \in \mathring{\mathsf{L}}_2$ shows that (m_3, n_3) (resp. (m_4, n_4)) is contained in region A (resp. E) of Figure 6.19 or vice-versa, and in both cases, the intersection p would not exist.

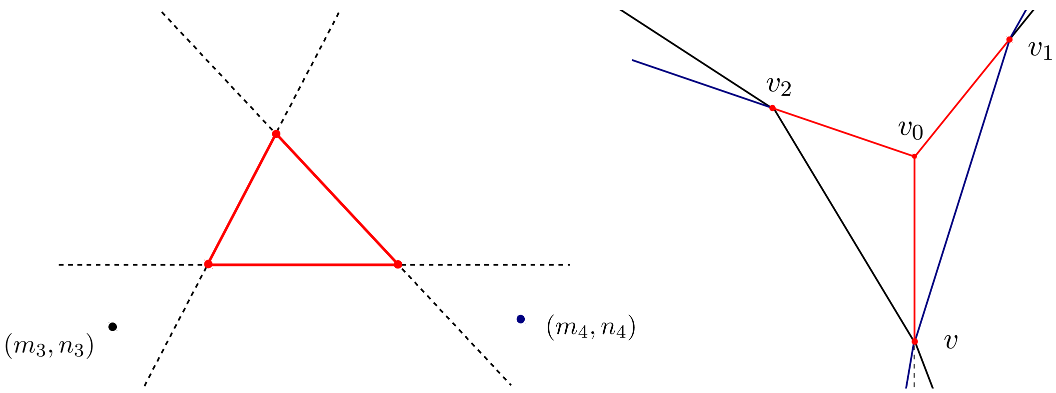

Figure 6.41: With $(m_3, n_3) \in A$ and $(m_4, n_4) \in E$, we have that T_1 and T_2 cannot intersect transversally.

6.7.1.2 There does not exist an intersection point of type (III)

Then there exists at most *two* (resp. *three*) transversal (resp. non-transversal) intersection points (resp. of type (II)) and together with v_0, this makes at most *six* positive solutions of (6.7.1).

6.7.2 The case $\alpha < 0 < \beta$

There does not exist a non-transversal intersection point at the origin of $\mathcal{E}$. To prove Theorem 6.61, we distinguish two cases.

6.7.2.1 There exists a non-transversal intersection of type (III)

There can be at most *three* non-transversal intersection points of type (II) (see Figure 6.42 on the left for an example) and at most *one* transversal intersection (c.f. Lemma 6.44). Without loss of generality, we may assume that the non-transversal intersection point of type (III) v is contained in L_0. Assume that (6.7.1) has more than *six* positive solutions, we prove that this gives a contradiction. The only way to have more than six positive solutions is to have seven ones such that their valuations are distributed in the folloing way. *Three* positive solutions with valuation $v \in \mathsf{L}_0$, *one* positive solution with valuation a transversal intersection point $p \in \mathsf{C}_2$ (by Lemma 6.44 since $v \in \mathsf{L}_0$) and *three* positive solutions where each has valuation a non-transversal intersection point of type (II).

The existence of such v and p means that T_2 has a vertex in L_0 and an edge in C_2, and since $\beta > 0$, we have that the point (m_4, n_4) is contained in the region A or E of Figure 6.19, say in E. Moreover, since T_1 and T_2 have *three* non-transversal intersection points of type (II) and $\alpha < 0$, the tropical curve T_1 has one vertex on each 1-cone of $\mathcal{E}$ (see Figure 6.42), and thus the point (m_3, n_3) is contained in the region G.

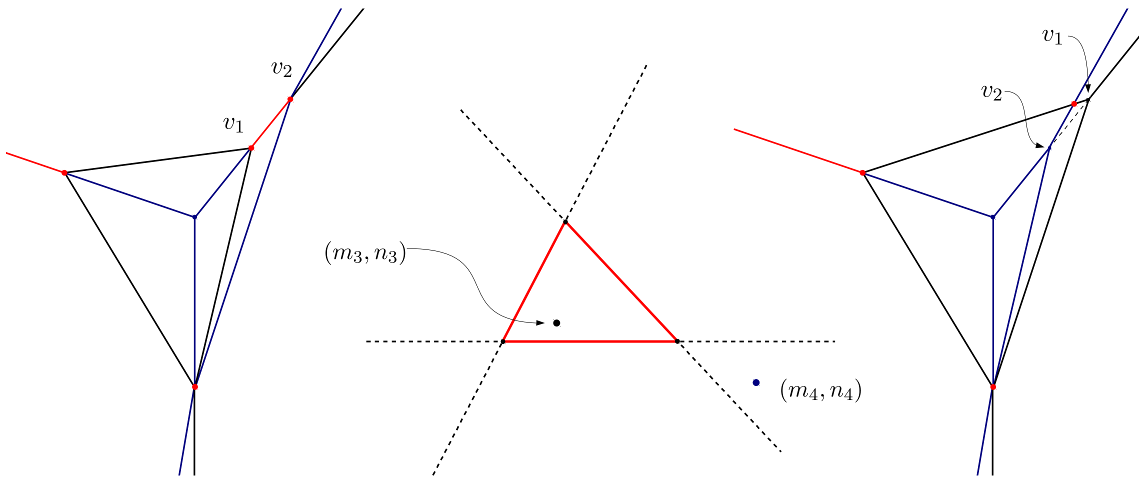

Figure 6.42: When $\alpha < 0 < \beta$, if T_1 intersects T_2 at five points of type (II), then the point (m_3, n_3) belongs to the triangle $[w_0, w_1, w_2]$.

Since $(m_3, n_3) \in G$ and $(m_4, n_4) \in E$, necessary conditions to have three non-transversal intersection points of type (II) is that the first coordinate of the vertex $v_1 \in \mathsf{L}_1$ of T_1 is less than the first coordinate of the vertex $v_2 \in \mathsf{L}_1$ of T_2 (see Figure 6.42). Indeed, otherwise there would only be one non-transversal intersection point of type (II) in L_2 (see Figure 6.42). However, if there exist two non-transversal intersection points of type (II) in L_1, then there does not exist a transversal intersection point in C_2 (see Figure 6.42 on the left). Conversely, if there exists a transversal intersection point in C_2, then there do not exist two non-transversal intersection points of type (II) in L_1 (see Figure 6.42 on the right). The incompatibility of these conditions gives the contradiction.

6.7.2.2 There does not exist an intersection point of type (III)

Since $\alpha < 0 < \beta$, the tropical curves T_1 and T_2 have respectively three and two vertices in the union of the 1-cones of $\mathcal{E}$. Therefore, there exists up to five non-transversal intersection points of type (II) and at most two transversal intersection points (see Figure 6.43). Using similar arguments to the case were there was an intersection of type (III), we deduce that the existence of five non-transversal intersection points of type (II) implies that there does not exist two transversal ones.

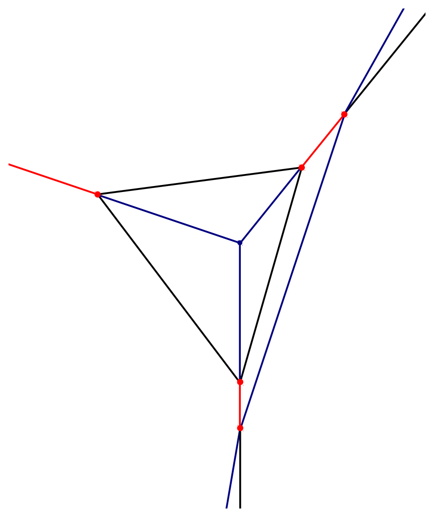

Figure 6.43: The curves T_1 and T_2 intersect in at most three non-transversal points of type (II).

6.7.3 The case $\alpha \leq \beta < 0$

There does not exist a non-transversal intersection point at the origin of $\mathcal{E}$. The proof of Theorem 6.61 comes easily whether there exists or not a non-transversal intersection point v of type (III). Indeed, if there exists v which is the valuation of at most three positive solutions of (6.7.1), then there exists at most two non-transversal intersection points of type (II) and at most one transversal intersection point (Lemma 6.44). Otherwise, the number of transversal and non-transversal of type (II) intersection points is at most two and three respectively.

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Introduction (en Français)

L'un des problèmes fondamentaux en mathématiques est de résoudre des équations polynomiales réelles puisque les systèmes polynomiaux apparaissent naturellement et de manière omniprésente en mathématiques et dans beaucoup de ses applications. On les voit apparaitre dans des domaines tels que la théorie du contrôle [Byr89], cinématique [BR90], chimie [GH02, MFR$^+$16] et beaucoup d'autres où c'est principalement les solutions réelles qui comptent. Dans cette introduction, nous donnons un bref aperçu sur la résolution des équations polynomiales et nous précisons les résultats principaux de cette thèse. Pour un exposé plus détaillé sur la résolution des équations polynomiales, voir par exemple [Sot11] ou [Stu02].

7.1 Polynômes en une variable

La théorie de Galois montre que pour un polynôme f à une variable en coefficients réels et degré inférieur ou égal à quatre, il existe une formule générale qui détermine explicitement les racines complexes de f en fonction de ses coefficients. Toutefois, cette affirmation est fausse si f a un degré supérieur à quatre. Cela signifie que le calcul des racines des polynômes en degré élevé n'est pas une tâche facile. Néanmoins, il existe de nombreuses méthodes et des résultats consacrés en particulier à ce problème (voir par exemple [Stu02]). Selon le *Théorème fondamental d'algèbre*, tout polynôme f en une variable admet au moins une racine complexe. En outre, le nombre de ses racines complexes (comptés avec multiplicités) est égale à son degré.

Malheureusement, le degré en général n'est pas la meilleure estimation du nombre de racines réelles de f, par exemple $1 - x^{100}$ admet 98 racines non réelles et seulement deux réelles. La règle de Descartes [Des97], qui remonte à 1637, est l'un des premiers résultats qui donne une estimation plus précise du nombre de racines réelles de f. Écrivons les termes de f en respectant l'ordre croissant de leurs exposants,

$$f(x) = b_0x^{k_0} + b_1x^{k_1} + \cdots + b_mx^{k_m}, \tag{7.1.1}$$

où $b_i \neq 0$ et $k_0 < \cdots < k_m$.

Théorème 7.1 (Règle de Descartes). *Le nombre r de racines positives isolées de f, comptées avec multiplicités, est au plus le nombre de changements de signe de ses coefficients,*

$$r \leq \{i \mid 1 \leq i \leq m \quad \textit{and} \quad b_{i-1}b_i < 0\}.$$

Théorème 7.1 est toujours vrai pour les polynômes en une variable avec des exposants réels. La conséquence immédiate de cette règle est que le nombre de solutions positives de f est majoré par m. En outre, en remplaçant x par $-x$ et en appliquant Théorème 7.1 au polynôme obtenu donne une estimation similaire pour le nombre de racines négatives de f. Par conséquent, le nombre de racines réelles non nulles de f est inférieur ou égal à $2m$.

Il est important de noter que la règle de Descartes, et donc la borne qui en résulte, est indépendante du degré. Cela amène naturellement à la question de généraliser Théorème 7.1 pour un système polynomial.

7.2 Systèmes polynomiaux creux

Considérons un système polynomial réel

$$f_1(z_1, \ldots, z_n) = \cdots = f_n(z_1, \ldots, z_n) = 0. \tag{7.2.1}$$

En général, nous cherchons des solutions de (7.2.1) dans le tore complexe $(\mathbb{C}^*)^n$ puisque les solutions dans les hyperplans de coordonnées sont des solutions dans des tores complexes de plus petites dimensions de systèmes tronqués. Une solution ζ de (7.2.1) est **non dégénérée** si les différentielles en ζ des fonctions définissant le système sont linéairement indépendantes. Les solutions non dégénérées sont plus faciles à manipuler puisque leur nombre ne diminuera pas après "petite" perturbation des coefficients du système associé.

7.2.1 Bornes polyédrales

Notons d_i le degré de f_i. Le Théorème fondamental de Bézout [Béz79] affirme que le nombre de solutions complexes non dégénérées de (7.2.2) est inférieur ou égal à $d_1 \cdots d_n$. En outre, cette borne est exacte. Les systèmes polynomiaux qui se produisent naturellement peuvent avoir une structure particulière, par exemple en termes de disposition des vecteurs d'exposants ou leur nombre (voir [Sot11]). Cependant, une grande partie de ces données combinatoires est négligée lors de l'utilisation du degré pour majorer le nombre de solutions complexes, et donc la borne de Bézout peut être grossière. En effet, il existe des bornes qui dépendent de la structure polyédrale associée au système polynomial.

À tout $w = (w^1, \ldots, w^n) \in \mathbb{Z}^n$, on associe un monôme $z^w \in \mathbb{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Considérons un polynôme de Laurent $f \in \mathbb{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ qui s'écrit ainsi

$$f(z) := \sum_{w \in \mathcal{W}} c_w z^w, \tag{7.2.2}$$

où $c_w \neq 0$ pour tout $w \in \mathcal{W}$. L'ensemble $\mathcal{W}$ est appelé le **support** de f. Le support d'un système (7.2.1) est l'union des supports de $f_1, \ldots, f_n$. Le **polytope de Newton** de f est l'enveloppe convexe $\Delta_{\mathcal{W}}$ de $\mathcal{W}$. Notons par $\mathrm{Vol}(\Delta)$ le volume Euclidien d'un polytope $\Delta \subset \mathbb{R}^n$. Nous avons le résultat fondamental suivant dû à A. Kushnirenko [Kus75].

Théorème 7.2 (Kushnirenko). *Si* (7.2.1) *admet $\mathcal{W}$ pour support, alors il a au plus $n!\,\mathrm{Vol}(\Delta_{\mathcal{W}})$ solutions isolées dans $(\mathbb{C}^*)^n$, et exactement ce nombre si* (7.2.1) *est générique parmi les systèmes de support $\mathcal{W}$.*

D. N. Bernstein [Ber75] affina ce résultat en prenant les supports individuels en compte. Désignons par $\mathcal{W}_i$ le support du polynôme f_i apparaissant dans (7.2.1). La **somme de Minkowski** des enveloppes convexes des $\mathcal{W}_i$ pour $i = 1, \dots, n$, est la somme

$$\Delta_{\mathcal{W}_1} + \cdots + \Delta_{\mathcal{W}_n} = \{w_1 + \cdots + w_n \mid w_1 \in \Delta_{\mathcal{W}_1}, \dots, w_n \in \Delta_{\mathcal{W}_n}\}.$$

Minkowski (voir [Ewa12]) a montré qu'étant donnés des objets convexes $K_1, \dots, K_n$ dans $\mathbb{R}^n$ et des nombres positifs $\lambda_1, \dots, \lambda_n$, la fonction $\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ est un polynôme homogène en $\lambda_1, \dots, \lambda_n$ de degré n. Donc il existe des coefficients $V(K_{i_1}, \dots, K_{i_n})$ pour $i_1, \dots, i_n \in [n]$ tels que

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1,\dots,i_n \in [n]} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{7.2.3}$$

Le **volume mixte** $\mathrm{MV}(K_1, \dots, K_n)$ de $K_1, \dots, K_n$ est égal à $V(K_1, \dots, K_n)$. On donne maintenant la généralisation faite par Bernstein du Théorème de Kushnirenko.

Théorème 7.3 (Bernstein)**.** *Un système de n polynômes en n variables dont les supports sont $\mathcal{W}_1, \dots, \mathcal{W}_n$ admet au plus $MV(\Delta_{\mathcal{W}_1}, \dots, \Delta_{\mathcal{W}_n})$ solutions isolées dans $(\mathbb{C}^*)^n$, et exactement ce nombre lorsque les polynômes sont génériques pour leurs supports donnés.*

Il est important de noter qu'une solution non dégénérée d'un système est une solution isolée. Les théorèmes de Kuschnirenko et de Bernstein donnent des majorations optimales pour le nombre de solutions non-dégénérées dans $(\mathbb{C}^*)^n$ d'un système polynomial. Bien que le degré et les bornes polyédrales précédentes sont aussi valables pour le nombre de solutions non-dégénérées dans $(\mathbb{R}^*)^n$, les bornes résultantes ne sont pas toujours optimales. Cela se produit généralement lorsque le support total $\mathcal{W}$ de (7.2.1) admet peu d'éléments relativement à $\Delta_{\mathcal{W}} \cap \mathbb{Z}^n$.

7.2.2 Bornes Fewnomiales

Notons par $\mathcal{W} \subset \mathbb{R}^n$ le support de (7.2.1). Les généralisations multivariées de la borne de Descartes (Théorème 7.1) pour les systèmes polynomiaux multivariés sont appelés **bornes Fewnomiales**[1]. Une attention particulière est portée aux solutions positives de (7.2.1), qui sont les solutions contenues dans l'orthant positif de $\mathbb{R}^n$. En effet, supposons qu'il existe une borne supérieure optimale $N_{\mathcal{W}}$ sur le nombre de solutions positives non dégénérées de (7.2.1) qui ne dépend que de $\mathcal{W}$. Alors $N_{\mathcal{W}}$ majore aussi le nombre de solutions contenus dans tout autre orthant, et donc (7.2.1) n'aura pas plus que $2^n N_{\mathcal{W}}$ solutions dans $(\mathbb{R}^*)^n$. Rappelons que Descartes a montré que nous avons $N_{\mathcal{W}} = |\mathcal{W}| - 1$ pour $n = 1$, mais encore, avant le livre de Khovanskii [Kho91], ce n'était pas clair qu'un tel $N_{\mathcal{W}}$ existe pour $n \geq 2$.

Théorème 7.4 (Khovanskii)**.** *Un système de n polynômes réels en n variables et comprenant $n + k + 1$ monômes distincts a moins que*

$$2^{\binom{n+k}{2}} (n+1)^{n+k}. \tag{7.2.4}$$

solutions positives non dégénérées.

L'existence d'une borne sur le nombre de solutions positives non dégénérées qui est indépendante des degrés des polynômes était révolutionnaire et est le point central du résultat de Khovanskii.

[1]Le terme "Fewnomial" a été inventé par A. Kushnirenko, où il a remplacé le terme "poly" du mot "polynomial", par le terme "Few" (voir [Kus08])

Elle confirme également le principe de Kushnirenko que la complexité topologique d'objets définis par des polynômes à coefficients réels peut être contrôlé par la complexité de la définition de ces polynômes plutôt que par les degrés ou polyèdres de Newton associés aux équations.

En outre, la borne du Théorème 7.4 n'est pas optimale. En fait Théorème 7.4 est un cas particulier d'un résultat plus général de Khovanskii concernant des solutions dans $\mathbb{R}^n$ de fonctions polynomiales en logarithmes des coordonnées et des monômes (voir [Kho91]). Par exemple, lorsque $k = 0$, le support $\mathcal{W}$ du système est un simplexe, et il y aura au plus *une* solution réelle. Bien qu'il ait été communément admis que la borne de Khovanskii (7.2.4) était loin d'être optimale, il s'avère que la tâche d'améliorer cette borne n'est pas facile.

La théorie des Fewnomials a été principalement initiée par la célèbre conjecture de Kushnirenko qui a été formulée à la fin des années soixante-dix comme une tentative de généraliser la borne de Descartes.

Conjecture 7.1 (Kushnirenko)**.** *Un système de n polynômes réels en n variables, dont les polynômes ont supports $\mathcal{W}_1, \ldots, \mathcal{W}_n$, admet au plus*

$$\prod_{i=1}^{n}(|\mathcal{W}_i| - 1)$$

solutions positives non dégénérées.

Ce n'est pas une tâche difficile de construire des systèmes polynomiaux atteignant la borne conjecturée par Kushnirenko. Notamment, une telle construction pourrait être par exemple un système

$$g_i(z_i) = 0, \quad \text{pour} \quad i = 1, \ldots, n$$

comprenant des polynômes en une variable, où chaque g_i admet m_i termes et $m_i - 1$ solutions positives non dégénérées (borne de Descartes). En fait, le manque de méthodes de construction efficaces a probablement incité Kushnirenko à établir sa conjecture.

7.3 Résultats avant la thèse

Après le fameux Théorème de Khovanskii, de nombreuses contributions récentes consacrées à la théorie des Fewnomials ont eu lieu, (voir [Sot11] pour une enquête). Dans cette section, nous donnons juste quelques résultats parmi des nombreux autres développés dans ce millénaire. La plupart de ces résultats seront ensuite étudiés et dans certains cas améliorés dans cette thèse.

7.3.1 Autour de la borne de Khovanskii

Considérons un système polynomial réel

$$f_1(z) = \cdots = f_n(z) = 0 \tag{7.3.1}$$

en n variables, supporté par un ensemble $\mathcal{W} \subset \mathbb{Z}^n$ tel que $|\mathcal{W}| = n + k + 1$ pour un certain $k \geq 1$. Dans [BS07], F. Bihan et F. Sottile ont réduit de manière significative la borne fewnomiale de Khovanskii (7.2.4) en montrant qu'il y a moins de

$$\frac{e^2 + 3}{4} 2^{\binom{k}{2}} n^k \tag{7.3.2}$$

solutions positives non dégénérées de (7.3.1). La méthode qu'ils utilisaient consiste à réduire le système de départ en un système de k équations en k variables, appelé le *transformé de Gale*. Ce transformé de Gale dépend de la configuration des vecteurs “Gale” duale aux exposants des monômes dans le système original (voir [BS08]). Cette réduction donne que la borne supérieure de la transformée de Gale est également vraie pour le nombre de solutions de (7.3.1). La borne dans (7.3.2) est également vraie pour les polynômes avec des exposants réels. En outre, (7.3.2) est asymptotiquement optimale dans le sens qu'en fixant k, il existe des systèmes avec $O(n^k)$ solutions positives [BRS08].

La constante $\frac{e^2+3}{4}$ qui apparait dans (7.3.2) est artificielle, son but est seulement de majorer une expression plus compliquée. En outre, les auteurs de [BS07] estiment que le terme $2^{\binom{k}{2}}$ dans (7.3.2) est considérablement exagérée. La borne dans (7.3.2) est également vraie pour les polynômes avec des exposants réels. Notons que lorsqu'on pose $n = k = 2$ dans (7.2.4), on obtient $2^6 \cdot 3^4 = 5184$. Bien que la nouvelle borne 15 est une borne fewnomiale considérablement plus petite pour un système avec $n = k = 2$, les auteurs de [BS07] affirment que la borne optimale est encore plus petite. Le cas $n = k = 2$ est le premier cas où nous ne savons pas grand-chose. En fait, avant cette thèse, la première construction connue, donnant beaucoup de solutions positives non dégénérées d'un système de deux polynômes à deux variables avec cinq monômes était essentiellement celle de B. Haas (7.3.5). Une telle construction donne cinq solutions positives non dégénérées, et montre que la borne supérieure optimale sur le nombre de solutions positives non dégénérées est supérieure ou égale à 5. Dans ce qui suit, nous appellerons un système de deux équations à deux variables avec cinq monômes distincts un système de type $n = k = 2$.

7.3.2 Utilisation du patchwork combinatoire

Considérons un système

$$f_{1,t}(z) = \cdots = f_{n,t}(z) = 0, \tag{7.3.3}$$

où chaque polynôme est obtenu à partir d'un polynôme $\sum_w c_w z^w$ de (7.3.1) en multipliant chaque monôme $c_w z^w$ par une puissance réelle de t, où t est un paramètre positif qui sera pris très proche de zéro. Soit $V(f_{i,t})$ l'ensemble des zéros de $f_{i,t}$ dans $\mathbb{R}^n$. Pour tout $\epsilon \in \{\pm 1\}^n$, considérons l'orthant

$$(\mathbb{R}_{>0})^\epsilon := \{x \in \mathbb{R}^n \mid x_i \epsilon_i > 0 \quad i = 1, \ldots, n\},$$

et soit $V_\epsilon(f_{i,t})$ l'intersection de $V(f_{i,t})$ avec $(\mathbb{R}_{>0})^\epsilon$.

Le Théorème de O. Viro affirme qu'on peut construire combinatoirement à la fois un espace Q_ϵ et un complexe simplicial $\mathcal{Z}_\epsilon \subset Q_\epsilon$ tel que le couple $(Q_\epsilon,\ \mathcal{Z}_\epsilon)$ est homéomorphe à $((\mathbb{R}_{>0})^\epsilon,\ V_\epsilon(f_{i,t}))$ pour $t > 0$ suffisamment petit. A partir de cela, on peut récupérer (à homéomorphismes près) toute l'hypersurface $V(f_{i,t})$ (pour $t > 0$ suffisamment petit) en recollant à la fois ses différentes parties, et leurs espaces ambiants.

Cela été généralisé par B. Sturmfels [Stu94] pour toute intersection complète $V(f_{1,t}) \cap \cdots \cap V(f_{s,t})$, avec $s \leq n$, étant donné que les exposants de t sont “suffisamment génériques”. Lorsque $s = n$, cette méthode peut être utilisée pour construire des systèmes avec un beaucoup de solutions positives non dégénérées et supports données. Récemment, F. Bihan [Bih14] a donné une borne supérieure sur le nombre de solutions réelles non-dégénérées qui sont construits en utilisant la généralisation de Sturmfels du Théorème de Viro. Cette borne est obtenue en utilisant le *volume mixte discret* des supports des $f_{i,t}$. De plus, il a démontré que cette borne est plus petite que celle donnée dans la conjecture de Kushnirenko (voir Sous-section 7). Lorsque $n = 2$ et $k = 1$, le volume mixte discret n'est pas plus grand que 3 et la borne correspondante est optimale (voir Sous-section 7). Lorsque $n = k = 2$, c'est facile de déduire par calcul que le volume mixte discret n'est pas plus grand que 6 (voir Lemme 6.4 dans le Chapitre 6), et ce n'est pas connu si la borne correspondante est optimale.

7.3.3 Systèmes supportés sur des circuits

L'un des premiers cas non-triviaux apparait lorsque $n \geq 2$ et $k = 1$, et dans ce cas là, le support $\mathcal{W}$ de (7.3.1) est un ensemble de $n + 2$ points dans $\mathbb{R}^n$. F. Bihan [Bih07] a démontré que chaque système polynomial supporté par tel $\mathcal{W}$ admet au plus $n + 1$ solutions positives non-dégénérées et que cette borne est optimale. En outre, si cette borne est atteinte, alors $\mathcal{W}$ est minimalement affinement dépendent, qui signifie que c'est un *circuit* dans $\mathbb{R}^n$. Les systèmes polynomiaux supportés par un circuit dans $\mathbb{Z}^n$ dont toutes les solutions complexes non dégénérées sont positives ont été étudiés dans [Bih15] (un tel système est appelé *maximallement positif*). Comme résultat principal, il est donné pour tout entier positif n une liste finie des circuits dans $\mathbb{Z}^n$ qui peuvent supporter des systèmes maximalement positifs à une action du groupe des transformations affines inversibles de $\mathbb{Z}^n$ près.

F. Bihan et A. Dickenstein [BD16] ont présenté la première version multivariée de la règle de Descartes pour borner le nombre des solutions positives réelles non dégénérées d'un système supporté par un circuit, en fonction de la variation de signe d'une suite associé aux vecteurs d'exposants et aux coefficients donnés. Il est aussi démontré que la borne obtenue est optimale et est reliée à la signature du circuit.

La première fois que les *dessins d'enfant réels* de Grothendieck, qui sont des graphes immergés dans la sphère de Riemann, ont été utilisés dans le contexte fewnomials est due à F. Bihan [Bih07]. Notamment, il utilise des dessins d'enfant pour montrer l'exactitude de la borne $n + 1$ pour le nombre de solutions positives d'un système supporté par un circuit $\mathcal{W} \subset \mathbb{R}^n$. Il a aussi démontré en utilisant la même technique, l'optimalité de cette borne pour le nombre des solutions *réelles* de ces systèmes. Il se trouve que, si l'on peut réduire un système fewnomial à une fonction polynomiale rationnelle $\mathbb{C}P^1 \to \mathbb{C}P^1$, alors on peut espérer d'utiliser les dessins d'enfant réels d'une manière fructueuse afin d'étudier de près le système original. Cette technique donne un point de vue intéressant sur la construction de systèmes polynomiaux avec un grand nombre de solutions réelles (voir Chapitre 3), la caractérisation de tels systèmes (voir Chapitre 5) et même majorer le nombre de solutions positives de systèmes polynomiaux creux (voir Chapitre 4).

La version de Sturmfels du *patchwork combinatoire* de Viro est encore une autre technique efficace de la géométrie algébrique réelle qui peut être utilisée pour construire des systèmes polynomiaux avec beaucoup de solutions réelles. Cette généralisation [Stu94] est pour les intersections complètes des hypersurfaces algébriques réelles. Parmi beaucoup d'autres utilisations dans le contexte des Fewnomials, citons le papier de K. Phillipson et J.-M. Rojas [PR13] où il est construit des systèmes polynomiaux supportés par un circuit dans $\mathbb{Z}^n$ et avec $n + 1$ solutions positives non dégénérés dans le cas de corps de base autres que $\mathbb{R}$.

7.3.4 Autour de la conjecture de Kuschnirenko

Considérons un système (7.3.1), et pour $i = 1, \ldots, n$, notons par m_i le nombre de points contenus dans le support de f_i. Rappelons que la Conjecture de Kushnirenko 7.1 affirme que (7.3.1) ne peut pas avoir plus de

$$\prod_{i=1}^{n}(m_i - 1)$$

solutions positives non dégénérées.

7.3.4.1 Premiers contre-exemples

La borne conjecturée n'est pas une borne sur le nombre de solutions positives isolées. W. Fulton donna le contre-exemple suivant dans [Ful13] (voir aussi [Stu02]). Considérons le système

$$\prod_{i=1}^{m}(z_1 - i)^2 + \prod_{i=1}^{m}(z_2 - i)^2 = 0, \quad z_1(z_3 - 1) = 0, \quad z_2(z_3 - 1) = 0, \tag{7.3.4}$$

où $m \geq 5$. La Conjecture de Kushnirenko prédit qu'un tel système admet au plus $(4m+1-1)(2-1)(2-1) = 4m$ solutions positives réelles. Cependant, il y a m^2 solutions positives de (7.3.4) de la forme $(i, j, 1)$, pour $i, j \in \mathbb{N}^*$ entre 1 et m.

Un cas particulier de la Conjecture de Kuchnirenko affirme que lorsque $n = 2$ et $m_1 = m_2 = 3$, le système (7.3.1) admet au plus quatre solutions positives non dégénérées. Dans un effort pour réfuter cette conjecture, Haas montra dans [Haa02] que

$$10x^{106} + 11y^{53} - 11y = 10y^{106} + 11x^{53} - 11x = 0 \tag{7.3.5}$$

admet cinq solutions positives non dégénérées. Bien avant, Konstantin A. Sevastyanov, un collègue de Kushnirenko, a trouvé un contre-exemple similaire. Malheureusement, ce contre-exemple ne semble pas avoir été retrouvé et, tragiquement, Sevastyanov est mort avant la publication de son contre-exemple.

Il a été montré après dans [LRW03], en utilisant une analyse au cas-par-cas, que lorsque $n = 2$ et $m_1 = m_2 = 3$, la borne supérieure optimale sur le nombre de solutions positives non dégénérées est cinq. En outre, il est démontré dans le même papier que si cette borne est atteinte, la somme de Minkowski des polytopes de Newton Δ_1 et Δ_2 associés est un hexagone.

Un système polynomial plus simple

$$x^6 + (44/31)y^3 - y = y^6 + (44/31)x^3 - x = 0, \tag{7.3.6}$$

qui aussi admet cinq solutions réelles positives non dégénérées a été découvert par A. Dickenstein, J.-M. Rojas, K. Rusek et J. Shih [DRR07]. De plus, ils ont montré que tels systèmes sont rares dans le sens suivant. Ils étudient la variété discriminant des espaces des coefficients du système polynomial

$$x^{2d} + ay^d - y = y^{2d} + bx^d - x = 0, \tag{7.3.7}$$

avec les paramètres (a, b, d), et montrent que les chambres (composantes connexes du complémentaire) contenant les systèmes avec le nombres maximal de solutions positive sont "petites".

7.3.4.2 Un trinôme et un t-nôme

Les systèmes polynomiaux réels en deux variables

$$f = g = 0, \tag{7.3.8}$$

où f admet $t \geq 3$ termes non-nuls et g admet trois termes non-nuls ont été étudiés par T.Y. Li, J.-M. Rojas and X. Wang [LRW03]. Ils ont démontré qu'un tel système, en permettant des exposants réels, admet au plus $2^t - 2$ solutions positives isolées. L'idée est de substituer une variable du t-nôme en fonction de l'autre, et de réduire le système à une fonction analytique en une variable

$$h(x) = \sum_{i=1}^{t} a_i x^{k_i}(1-x)^{l_i},$$

où tous les coefficients et exposants sont des réels. Le nombre de solutions positives de (7.3.8) est égal au nombre de solutions de $h = 0$ contenues dans $]0, 1[$. Les techniques principales utilisées dans [LRW03] sont une extension du Theorème de Rolle et une récurrence qui comprend des dérivées de certaines fonctions analytiques. En fait, les résultats de Li, Rojas et Wang [LRW03] sont plus généraux. Considérons un système polynomial

$$f_1 = \cdots = f_n = 0 \tag{7.3.9}$$

à n variables, où les fonctions $f_1, \ldots, f_{n-1}$ sont des trinômes et f_n admet t monômes distincts. Les auteurs dans [LRW03] montrent que (7.3.9) admet au plus $n + n^2 + \cdots + n^{t-1}$ solutions positives non dégénérées.

La borne exponentielle $2^t - 2$ sur le nombre de solutions positives de (7.3.8) a été récemment raffinée par P. Koiran, N. Portier et S. Tavenas [KPT15b] en une borne polynomiale. Ils ont considéré une fonction analytique en une variable

$$\sum_{i=1}^{t} \prod_{j=1}^{m} f_j^{\alpha_{i,j}}, \tag{7.3.10}$$

où tous les f_j sont des polynômes réels de degrés au plus d et tous les exposants de f_j sont réels. En utilisant les Wronskians des fonctions analytiques, il a été démontré que le nombre de solutions positives de (7.3.10) dans un intervalle I (en supposant que $f_j(I) \subset]0, +\infty[$) est majoré par $\frac{t^3 md}{3} + 2tmd + t$. Comme cas particulier (en considérant $m = 2$, $d = 1$ et $I =]0, 1[$), ils obtiennent que $h(x) = \sum_{j=1}^{t} a_i x^{k_i}(1-x)^{l_i}$ admet au plus $2t^3/3 + 5t$ racines dans I.

7.3.4.3 Une courbe plane et une droite

Lorsque le trinôme g de (7.3.8) est un polynôme de degré un, la borne optimale sur le nombre de solutions réelles non-dégénérées de (7.3.8) est une fonction linéaire en t.

Notamment, M. Avendaño montra dans [Ave09] que si un tel système n'admet pas un nombre infini de solutions réelles, il admet au plus $6t - 6$ solutions dans $(\mathbb{R}^*)^2$, comptés avec multiplicités. En particulier, il a démontré que le nombre de solutions *positives* non dégénérées de (7.3.8) est au plus $2t - 2$. La méthode utilisée dans [Ave09] consiste à remplacer z_2 par $az_1 + b$ dans (7.3.8) pour certains réels non-nuls a et b. De cette façon, avec l'aide de la règle de Descartes appliquée au polynôme en une variable qui en résulte, on obtient finalement la borne $2t - 2$.

7.3.5 Autour d'une conjecture polynomiale-fewnomiale

A. Kushnirenko formula aussi la conjecture suivante (pour plus de détails sur le sujet, voir [Kus08]). Considérons un système

$$f(x, y) = g(x, y) = 0 \tag{7.3.11}$$

de deux équations en deux variables, où g est un polynôme avec t monômes distincts, et f est un polynôme de degré d.

Conjecture 7.2. *Le système* (7.3.11) *admet au plus* $N(d, t)$ *solutions positives non dégénérées, où* $N(d, t)$ *est une fonction ne dépendant que des nombres* d *et* t.

Sevostyanov prouva en 1978 qu'une telle fonction $N(d, t)$ existe. Pourtant, ce résultat (avec son contre-exemple à la conjecture de Kushnirenko) ne fut jamais publié. Selon [Sot11], ce résultat fut une source d'inspiration pour Khovanskii pour développer la théorie des Fewnomials.

Évidemment, d'après les bornes de Khovanskii et Bihan-Sottile, une telle fonction $N(d, t)$ existe, néanmoins comme (7.3.11) est un cas très particulier d'un système générique (7.2.1), les bornes (7.2.4) et (7.3.2) (qui sont exponentielles en d et t) peuvent être trop larges. La borne de M. Avendaño [Ave09] montre que $N(1, t) \leq 2t - 2$, qui est en effet optimale au moins pour $t = 3$ (voir [BEH15]).

La plus petite borne inférieure jusqu'à présent pour toutes valeurs d et t à été découverte par P. Koiran, N. Portier et S. Tavenas [KPT15a]. Ils ont montré que (7.3.11) admet au plus $O(d^3t+d^2t^3)$ solutions réelles lorsque ce nombre est fini. De plus, si l'ensemble de solutions réelles est infini, il admet au plus $O(d^3t + d^2t^3)$ composantes connexes.

7.4 Résultats de la thèse

Nous divisons nos principaux résultats en quatre chapitres.

7.4.1 Chapitre 3: Intersection d'une courbe plane creuse avec une droite

Le chapitre 3 est un travail en commun avec F. Bihan [BEH15]. Considérons un système

$$f(x, y) = ax + b - y = 0, \tag{7.4.1}$$

où $f \in \mathbb{R}[x, y]$, admet t termes non nuls. Dans le chapitre 3, tous les solutions dans $(\mathbb{R}^*)^2$ sont comptées avec multiplicités. Cela revient à compter le nombre de racines réelles d'un polynôme $f(x, ax + b)$, où $a, b \in \mathbb{R}$ et $f \in \mathbb{R}[x, y]$ admet au plus t termes non nuls. M. Avendaño montra dans [Ave09, Théorème 1.1] que (7.4.1) admet au plus $6t - 4$ solutions réelles comptées avec multiplicités sauf pour les racine possibles 0 et $-b/a$. La question d'optimalité n'était pas abordé dans [Ave09] et cela fut la motivation du travail actuel. Nous montrons le résultat suivant.

Théorème 7.5. *Soit $f \in \mathbb{R}[x, y]$ un polynôme ayant au plus t termes non nuls et soit a, b deux nombres réels. On suppose que le polynôme $g(x) = f(x, ax + b)$ est non nul. Alors g admet au plus $6t - \mathbf{7}$ racines réelles comptées avec multiplicités sauf pour les racines éventuelles 0 et $-b/a$ qui sont comptés au plus une seule fois.*

Les méthodes de démonstration de ce dernier résultat sont élémentaires, et constituent d'une version raffinée de celles de [Ave09]. Cela pourrait ressembler à une petite amélioration du résultat principal de [Ave09]. En fait, ce raffinement est non trivial, et la borne du Théorème 7.5 est optimale au moins pour $t = 3$.

Théorème 7.6. *Le nombre maximal de points d'intersections réels d'une droite réelle avec une courbe plane réelle définie par un polynôme ayant trois termes non nuls est onze.*

Explicitement, la courbe réelle d'équation

$$-0.002404\, xy^{18} + 29\, x^6y^3 + x^3y = 0 \tag{7.4.2}$$

intersecte la droite réelle $y = x + 1$ en précisément onze points dans $\mathbb{R}^2$.

La stratégie pour construire cet exemple est d'abord de déduire de la preuve du Théorème 7.5 quelques conditions nécessaires sur les monômes de l'équation souhaitée. Ensuite, l'utilisation des dessins d'enfant de Grothendieck d'une manière nouvelle aide à tester la faisabilité de certains monômes, puisque cette méthode donne une représentation claire de la topologie du graphe de $x \mapsto f(x, x + 1)$. Finalement, des expérimentations sur un logiciel conduisent à une équation précise (7.4.2).

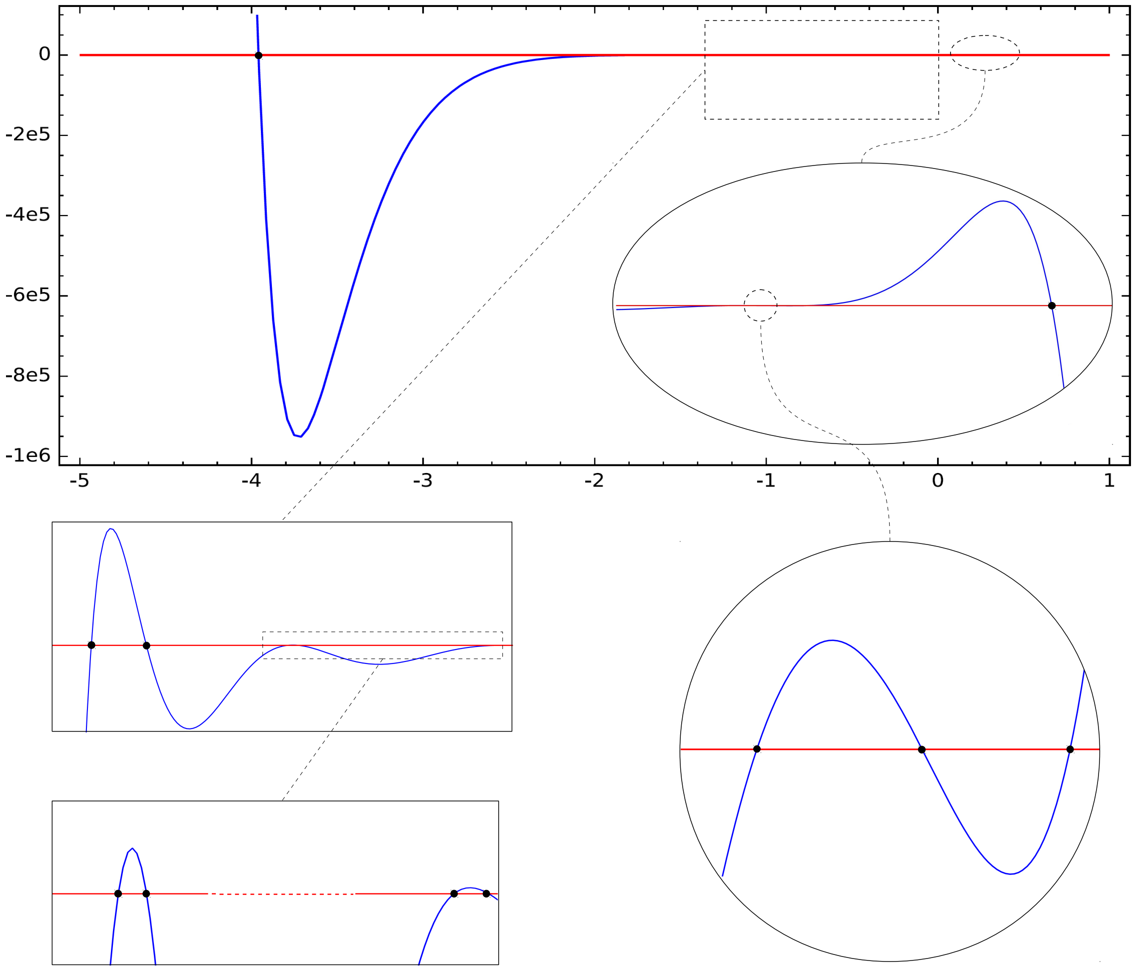

Figure 7.1: La courbe bleue représente le graphe de $x \mapsto f(x, x+1)$, et la droite rouge représente l'axe des abscisses (des parties de la courbe sont zoomées pour plus de clarté.)

7.4.2 Chapitre 4: Points d'intersection positifs d'une courbe trinomiale et d'une courbe t-nomiale

Considérons le système (7.3.8) où f admet $t \geq 3$ termes non nuls et g admet trois termes non nuls. Supposons que le dernier système admet un nombre fini de solutions. Soit $\mathcal{S}(3,t)$ dénote le nombre maximal de solutions positives non dégénérées de (7.3.8). On montre le résultat suivant dans la Section 4.2.

Théorème 7.7. *On a* $\mathcal{S}(3,t) \leq 3 \cdot 2^{t-2} - 1$.

Notons que puisque le nombre de solutions positives de deux trinômes en deux variables est borné par cinq (voir [LRW03]), la borne $\mathcal{S}(3,t)$ est optimale pour $t = 3$. En outre, pour $t = 4, \ldots, 9$, cette nouvelle borne est plus petite que les bornes $2^t - 2$ et $2t^3/3 + 5t$, obtenues dans [LRW03] et [KPT15b] respectivement, et montre par exemple que $6 \leq \mathcal{S}(3,4) \leq 11$.

Rappelons qu'en exprimant un variable du trinôme g de (7.3.8) en fonction de l'autre réduit le système à une fonction analytique en une variable

$$h(x) = \sum_{i=1}^{t} a_i x^{k_i} (1-x)^{l_i}.$$

Le nombre de solutions positives de (7.3.8) est égal à celui de $h = 0$ contenus dans $]0,1[$. On démontre le théorème 7.7 en utilisant la même approche que celle de [LRW03] i.e. on considère une récurrence faisant intervenir des dérivées de fonctions analytiques en une variable associées au système (7.3.8). En commençant avec la fonction $f_1 = h$, à chaque étape $1 < i < t$, on se retrouve avec une fonction f_i définie comme une certaine dérivée de f_{i-1} multipliée par des puissances de x et de $(1-x)$. En appliquant le Théorème de Rolle à chaque f_i, on peut borner le nombre de ses racines contenues dans $]0,1[$ en fonction des racines de f_{i-1} dans le même intervalle. Il apparaît que dans l'étape $t-2$, on est réduit à borner le nombre de solutions dans $]0,1[$ de l'équation $\phi(x) = 1$, où

$$\phi(x) = \frac{x^{\alpha}(1-x)^{\beta} P(x)}{Q(x)},$$

$\alpha, \beta \in \mathbb{Q}$, et à la fois P et Q sont des polynômes réels de degrés au plus $2^{t-2} - 1$.

La plus grande partie du Chapitre 4 est consacrée à la preuve dans la Section 4.3 du résultat suivant.

Théorème 7.8. *On a* $\sharp\{x \in]0,1[\ |\phi(x) = 1\} \ \leq \deg P + \deg Q + 2.$

En choisissant $m \in \mathbb{N}$ tel qu'à la fois $m\alpha$ et $m\beta$ soient des entiers, on obtient alors une fonction rationnelle $\varphi := \phi^m : \ \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$. Les images inverses de 0, 1, ∞ sont données par les racines de P, Q, $\varphi - 1$, ainsi que 0 et 1 (si $\alpha\beta \neq 0$). Ces images inverses son contenues dans le graphe $\Gamma := \varphi^{-1}(\mathbb{R}P^1) \subset \mathbb{C}P^1$, qui est un exemple d'un dessin d'enfant réel de Grothendieck. Beaucoup de restrictions sur la topologie du graphe de φ apparaissent explicitement comme des restrictions sur $\Gamma = \varphi^{-1}(\mathbb{R}P^1)$. Notamment, les points critiques de φ correspondent aux sommets de Γ. Le nombre de racines de $\varphi - 1$ dans $]0,1[$ est contrôlé par le nombre de certains types de points critiques de φ appelées points critiques *positifs utiles*. En faisant une analyse fine sur Γ, on borne le nombre de sommets correspondants à ces points critiques en fonction de $\deg P$ et $\deg Q$.

On considère dans la Section 4.4 le cas $t = 3$ i.e. le cas de deux trinômes en deux variables. Rappelons que lorsque le nombre maximal de solutions positifs est atteint, la somme de Minkowski $\Delta_1 + \Delta_2$ est un hexagone (voir [LRW03]). Du point de vue des éventails normaux, ça signifie que l'éventail normal de la somme de Minkowski $\Delta_1 + \Delta_2$, qui est le raffinement commun des éventails normaux de Δ_1 et Δ_2, admet six cônes 2-dimensionnels (et six cônes 1-dimensionnels). On donne des contraintes supplémentaires suivantes sur la somme de Minkowski de Δ_1 et Δ_2 lorsque (7.3.8) admet cinq solutions positives. On dit que Δ_1 et Δ_2 alternent si chaque cône 2-dimensionnel de l'éventail normal de Δ_1 contient un cône 1-dimensionnel de l'éventail normal de Δ_2 ayant seulement l'origine comme face commune. Une analyse plus fine de Γ dans le cas $t = 3$ nous permet d'obtenir le résultat suivant.

Théorème 7.9. *Si le système* (7.3.8) *admet 5 solutions positives, alors* Δ_1 *et* Δ_2 *n'alternent pas.*

Les triangles de Newton Δ_1 et Δ_2 n'alternent pas veux dire qu'il existe deux arêtes consécutives de $\Delta_1 + \Delta_2$ qui sont des translatés de deux arêtes consécutives de Δ_1 ou bien de Δ_2. Figure 7.2 illustre ce théorème pour le système (7.3.6), et on fournit un autre exemple dans la Section 4.4.

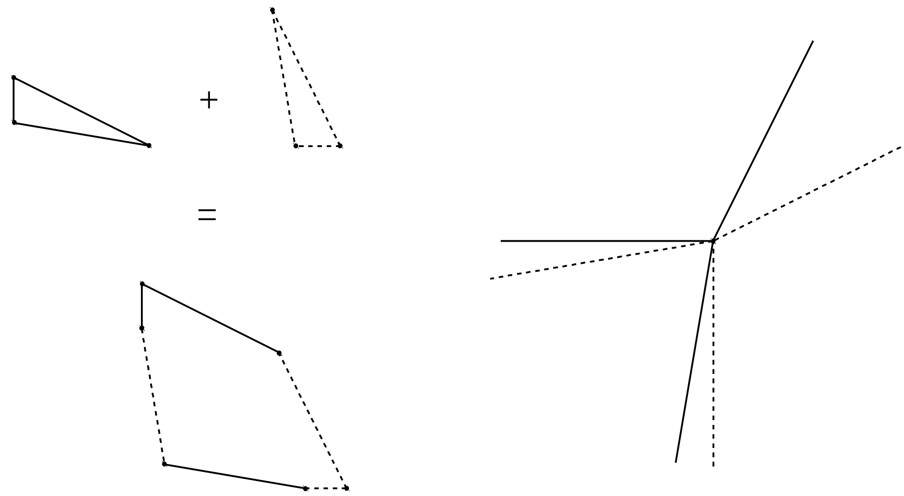

Figure 7.2: Les polytopes de Newton, leurs somme de Minkowski et les éventails normaux associées de (7.3.6).

7.4.3 Chapitre 5: Caractérisation des circuits supportant des systèmes polynomiaux avec le nombre maximal de solutions positives

Rappelons qu'un circuit $\mathcal{W} \subset \mathbb{R}^n$ est un ensemble de $n+2$ points distincts minimalement affinement dépendants. Une généralisation très récente de la règle de Descartes a été développée par F. Bihan et A. Dickenstein dans [BD16]. Ceci a donné des conditions sur à la fois le circuit et la matrice des coefficients qui sont nécessaires pour que le système admette $n+1$ solutions positives non dégénérées. Plus précisément, les auteurs de [BD16] montrent que si un tel système admet $n+1$ solutions positives non dégénérées, alors tous les mineurs maximaux de la matrice des coefficients sont non nuls et toute relation affine $\sum_{i=1}^{n+2} \lambda_i w_i = 0$ sur $\mathcal{W}$ admet le même nombre (à un écart de 1 si n est impair) de coefficients positifs que de coefficients négatifs. Dans le chapitre 5, on caractérise complètement les circuits qui supportent des systèmes polynomiaux ayant $n+1$ solutions positives non dégénérées.

Théorème 7.10. *Un circuit $\mathcal{W}$ dans $\mathbb{R}^n$ supporte un système avec $n+1$ solutions positives non dégénérées si et seulement si il existe une bijection*

$$\begin{array}{ccc} \{1,\ldots,n+2\} & \longrightarrow & \mathcal{W} \\ i & \longmapsto & w_i \end{array}$$

tel que chaque relation affine $\mathcal{W}$ peut s'écrire comme

$$\sum_{i=1}^{s} \alpha_i w_i = \sum_{s+1}^{n+2} \alpha_i w_i,$$

où $s = \lfloor (n+2)/2 \rfloor$ et tous les α_i sont des nombres positifs satisfaisant

$$\sum_{i=1}^{r} \alpha_i < \sum_{i=s+1}^{s+r} \alpha_i < \sum_{i=1}^{r+1} \alpha_i \quad \textit{pour} \quad r = 1,\ldots,s-1 \quad \textit{si} \quad n \quad \textit{est pair}$$

ou

$$\sum_{i=1}^{r}\alpha_i < \sum_{i=s+2}^{s+r+1}\alpha_i < \sum_{i=1}^{r+1}\alpha_i \quad \textit{pour} \quad r=1,\ldots,s-1 \quad \textit{si} \quad n \quad \textit{est impair.}$$

F. Bihan montra dans [Bih15] que si un circuit dans $\mathbb{Z}^n$ supporte un système maximalement positif avec $n+1$ solutions positives non dégénérées, alors ce circuit admet une relation affine primitive (i.e. relation affine avec des coefficients entiers premiers entre eux) comme celle dans le théorème 7.10 avec $\alpha_1 = \alpha_{n+2} = 1$ et tous les autres coefficients sont égaux à deux. Ceci peut être vu comme une conséquence du théorème 7.10 (voir Exemple 5.12, Section 5.2). En effet, si $\mathcal{W}$ supporte un système maximalement positif avec $n+1$ solutions positives non dégénérées, alors le sous-groupe de $\mathbb{Z}^n$ engendré par $\mathcal{W}$ est $\mathbb{Z}^n$. En outre, si $\sum_{i=1}^{s}\alpha_i w_i = \sum_{s+1}^{n+2}\alpha_i w_i$ est une relation affine primitive, alors $\sum_{i=1}^{s}\alpha_i = \sum_{s+1}^{n+2}\alpha_i = n+1$ (voir [Bih15] pour plus de détails), ce qui avec les inégalités du théorème 7.10 implique les égalités voulues. Afin de démontrer le théorème 7.10, on peut se ramener au cas où $\mathcal{W} \subset \mathbb{Z}^n$ (voir la première partie du Chapitre 5). On démontre la partie "seulement si" du théorème 7.10 de la façon suivante. Considérons un système polynomial supporté par un circuit en n équations à n variables qui admet le nombre maximal de solutions positives non dégénérées. On lui associe en utilisant la dualité de Gale (voir Section 5.1) une function à une variable

$$\varphi(y) = \prod_{i=1}^{n+1} P_i^{\lambda_i},$$

où P_i est un polynôme de degré 1 qui dépend des équations du système, $\sum_{i=1}^{n+2}\lambda_i(w_i - w_0) = 0$ est une relation linéaire entre les vecteurs $w_i - w_0$ et les solutions positives non dégénérées du système initial sont en bijection avec les solutions de $\varphi(y) = 1$ contenues dans

$$\Delta_+ = \{y \in \mathbb{R}_{>0} \mid P_i(y) > 0,\ i = 1,\ldots,n+1\}.$$

L'homogénisation de φ est une application rationnelle $\mathbb{C}P^1 \to \mathbb{C}P^1$, telle que l'image inverse de $\mathbb{R}P^1$ par cette homogénisation est le dessin d'enfant réel Γ (voir le chapitre 2). Comme les valences des sommets de Γ sont contrôlées par les entiers λ_i et les racines de P_i pour $i = 1,\ldots,n+1$, en analysant Γ, on obtient les inégalités du théorème 7.10.

Les solutions de $\varphi(y) = 1$ dans Δ_+ sont les racines du *polynôme de Gale*

$$G(y) = \prod_{\lambda_i>0} P_i^{\lambda_i}(y) - \prod_{\lambda_i<0} P_i^{-\lambda_i}(y) \tag{7.4.3}$$

dans le même intervalle. Dans [PR13, preuve du Lemme 1.8], K. Phillipson et J.-M. Rojas ont construit des systèmes polynomiaux supportés par un circuit dans $\mathbb{Z}^n$ avec $n+1$ solutions positives non dégénérées en utilisant les *polynômes de Viro* $P_{i,t}(y) = a_i + t^{\alpha_i} b_i$, où $a_i, b_i, \alpha_i \in \mathbb{R}$, et $t > 0$ est un paramètre qui seras pris suffisamment petit. Ils appliquent la version de Sturmfels du patchwork combinatoire de Viro développé dans [Stu94] qui comprend la subdivision mixte des polytopes de Newton. Ici, on utilise aussi les polynômes de Viro $P_{i,t}$, et on regarde directement les racines dans Δ_+ des polynômes de Gale correspondants. Les inégalités dans Théorème 7.10 apparaissent explicitement comme étant nécessaires pour construire des systèmes polynomiaux supportés par un circuit dans $\mathbb{Z}^n$ avec $n+1$ solutions positives non dégénérées en utilisant les polynômes de Viro $P_{i,t}$.

7.4.4 Chapitre 6: Construire des systèmes polynomiaux avec beaucoup de solutions positives

La *géométrie tropicale* est un nouveau domaine des mathématiques qui se situe à la croisée de domaines tels que la géométrie torique, la géométrie complexe ou réelle, et la combinatoire [Mik06, MR05, MS15]. Il se trouve que la généralisation de Sturmfels du Théorème de Viro peut être reformulée dans le contexte de la géométrie tropicale (voir [Mik04, Rul01]). Ce qui fait de la géométrie tropicale un outil effectif pour construire des systèmes polynomiaux avec un support prescrit et avec beaucoup de solutions positives.

Rappelons que la meilleure borne fewnomiale connue sur le nombre de solutions positives non dégénérées d'un système polynomial réel de n équations en n variables supporté par un ensemble de $n+k+1$ points où $k, n \geq 1$, est égale à $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$ [BS07]. En fait, le même papier contient la meilleure borne supérieure 15 lorsque $n = k = 2$. D'un autre côté, les meilleures constructions connues donnent 5 solutions positives non dégénérées (voir [Haa02]). La motivation derrière le chapitre 6 est d'utiliser la version de Sturmfels du patchwork combinatoire de Viro, et autres outils et résultats (voir Chapitre 2, Sous-section 2.2.6) développés dans la géométrie tropicale pour construire un système de deux équations en deux variables et avec cinq monômes en total (un système du type $n = k = 2$ en abrégé) ayant beaucoup de solutions positives.

Soit $\mathbb{K}$ le corps des **séries de Puiseux generalisées localement convergentes**

$$a(t) = \sum_{r \in R} \alpha_r t^r,$$

où $R \subset \mathbb{R}$ est un ensemble bien ordonné et $a(t)$ est une série complexe convergente pour $t > 0$ suffisamment petit. Ceci est un corps algébriquement clos. Considérons le sous-corps $\mathbb{RK}$ de $\mathbb{K}$ formés des séries de Puiseux généralisées *réelles*, qui veut dire que les α_r apparaissant dans $a(t)$ sont des nombres réels. On considère dans le chapitre 6 un système polynomial (de Laurent) creux

$$f_1(z) = f_2(z) = 0, \tag{7.4.4}$$

dont les équations sont définies sur $\mathbb{RK}$. On suppose que (7.4.4) admet un nombre fini de solutions, toutes non dégénérées. Un élément **positif** $a(t)$ de $\mathbb{K}$ est un élément de $\mathbb{RK}^*$ dont le coefficient du terme de premier ordre est positif.

À un polynôme de Laurent $f(z) = \sum_{w \in \mathcal{W}} c_w z^w \in \mathbb{R}[z]$, on associe un *polynôme tropical*

$$f_{\text{trop}}(x) = \text{“} \sum_{w \in \mathcal{W}} \text{val}(c_w) x^w \text{”},$$

où $\text{val}(c_w)$ est moins l'ordre (dans le sens classique) des séries de Puiseux c_w, et les opérations sont les opérations tropicales (la somme est le max, et le produit est la somme classique). L'*hypersurface tropicale* associée T est le lieux des coins de la fonction convexe linéaire par morceaux $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f_{\text{trop}}(x)$. Par le Théorème de Kapranov [Kap00] (voir Chapitre 2, Sous-section 2.2.2), l'hypersurface tropicale T coïncide avec la clôture de

$$\text{Val}\left(\{z \in (\mathbb{K}^*)^n \mid f(z) = 0\}\right),$$

où Val est l'extension de la fonction val coordonnée par coordonnée. La **partie positive** de T est la clôture de $\text{Val}\left(\{z \in (\mathbb{RK}_{>0})^n \mid f(z) = 0\}\right)$.

Considérons maintenant encore les polynômes $f_1, f_2 \in \mathbb{RK}[z_1^{\pm 1}, z_2^{\pm 1}]$ définissant deux courbes tropicales $T_1, T_2 \subset \mathbb{R}^2$. Supposons pour le moment que T_1 et T_2 s'intersectent transversalement, ce qui signifie que chaque point d'intersection est isolé et contenu dans l'intérieur relatif d'une pièce linéaire 1-dimensionnelle de T_1 et une autre pièce linéaire 1-dimensionnelle de T_2. Alors par la généralisation de Sturmfels du Théorème de Viro, chaque point d'intersection de T_1 et T_2 contenu dans les deux parties positives (point d'intersection positif en bref) se remonte à une unique solution de (7.4.4) dans $(\mathbb{R}\mathbb{K}_{>0})^2$, ce qui donne des solutions positives d'un système réel $g_1(z) = g_2(z) = 0$ en prenant $t > 0$ suffisamment petit. Rappelons que dans le cas où $n = k = 2$ (ce qui signifie que les équations de T_1 et T_2 ont en total cinq monômes), le nombre de points d'intersections transverses de T_1 et T_2 est majoré par six (voir Sous-section 7). On démontre que cette borne est optimale et peut être réalisée par des points d'intersections positifs.

Proposition 7.3. *Il existe deux courbes tropicales planes T_1 et T_2 définies par des équations ayant cinq monômes distincts au total et qui ont six points d'intersections transverses positifs.*

Par conséquent, en utilisant la généralisation de Sturmfels de la Théorème de Viro (comme expliqué au dessus), ceci donne un système de type $n = k = 2$ admettant six solutions positives non dégénérées. Afin d'obtenir un système de type $n = k = 2$ avec plus que six solutions positives non dégénérées, on considère donc des courbes tropicales T_1 et T_2 qui ne s'intersectent pas transversalement.

Notons que $T_1 \cap T_2$ est linéaire par morceaux et ses pièces linéaires sont soit des point isolés, soit des segments. Heureusement, si une pièce linéaire $\xi \subset T_1 \cap T_2$ est un point isolé, alors les résultats de [Kat09, Rab12, OP13] et [BLdM12] montrent que ξ se remonte en des solutions de (7.4.4) dans $(\mathbb{K}^*)^2$. Les solutions positives non dégénérées de (7.4.4) dont la valuation est égale à ξ peuvent être estimées en calculant le *système réduit* réel de (7.4.4) par rapport à ξ (voir Chapitre 2, Sous-section 2.2.6). Par contre, si cette pièce linéaire ξ a une dimension égale à 1, alors ξ est un ensemble infini contenant un ensemble fini (éventuellement vide) de points qui sont les valuations des solutions positives non dégénérées de (7.4.4). Ce n'est pas facile de localiser ces valuations. En fait, la seule méthode pour accomplir cette tâche, est appelée la *modification tropicale* (voir [Mik06, BLdM12]). Ce problème est traité dans la section 6.2 du chapitre 6 en utilisant une autre approche. Notamment, pour chaque pièce linéaire ξ de dimension 1, on associe un polynôme de Viro $f_{t,\xi}$ tel que tous les termes de premier ordre des solutions positives non dégénérées de (7.4.4) de valuation dans l'intérieur relatif de ξ peuvent être récupérés par le système réduit (7.4.4) par rapport à ξ et le polynôme de Viro $f_{t,\xi}$.

On considère maintenant le système (7.4.4) de type $n = k = 2$. Supposons qu'il n'existe pas une droite dans $\mathbb{R}^2$ contenant trois points du support du système. On montre dans la section 6.3 qu'on peut associer à ce système un nouveau système

$$\begin{aligned} a_0 + y_1^{m_1} + a_2 y_1^{m_2} y_2^{n_2} + a_3 t^{\alpha} y_1^{m_3} y_2^{n_3} &= 0, \\ b_0 + y_1^{m_1} + b_2 y_1^{m_2} y_2^{n_2} + b_4 t^{\beta} y_1^{m_4} y_2^{n_4} &= 0, \end{aligned} \tag{7.4.5}$$

dont les polynômes sont dans $\mathbb{R}\mathbb{K}[y_1^{\pm 1}, y_2^{\pm 1}]$, qui a le même nombre de solutions positives non dégénérées que (7.4.4), et satisfaisant que l'ordre de tous les a_i, b_j est nul, tous les m_i, n_i appartiennent à $\mathbb{Z}$ avec $m_1, n_2 > 0$, et α, β sont des nombres réels.

Les deux résultats principaux du chapitre 6 sont les suivants.

Théorème 7.11. *Si $(\alpha, \beta) \neq (0, 0)$, alors (7.4.5) admet au plus neuf solutions positives non dégénérées.*

Nous démontrons le théorème 7.11 dans la section 6.5. Notons que si $(\alpha, \beta) = (0, 0)$, alors on peut rien faire si on veut utiliser la géométrie tropicale. En effet, le problème de borner le nombre de solutions positives non dégénérées de (7.4.5) revient alors à borner le nombre de solutions positives d'un système polynomial réel de type $n = k = 2$.

Théorème 7.12. *Il existe un système* (7.4.5) *ayant sept solutions positives non dégénérées .*

La construction d'un système (7.4.5) qui admet sept solutions positives non dégénérées est effectué dans la section 6.5. Notamment, pour tout $0 < \alpha < \gamma_0$, le système

$$\begin{aligned} -1 + y_1^6 + y_1^3 y_2^6 - t^\alpha y_1^{-14} y_2^7 &= 0, \\ -1 + 0.36008 t^{\gamma_0} + y_1^6 + (1 - 0.36008 t^\alpha) y_1^3 y_2^6 - (44/31)^{\frac{5}{6}} t^\alpha y_1^{-12} y_2^9 &= 0, \end{aligned} \quad (7.4.6)$$

admet sept solutions positives non dégénérées.

On a effectué une analyse au cas par cas pour obtenir des conditions nécessaires pour que (7.4.5) admet plus que six solutions positives non dégénérées. En particulier, on a obtenu dans les Sections 6.6 et 6.7 le résultat suivant.

Théorème 7.13. *Si* $(\alpha, \beta) \neq (0,0)$, *et l'une des conditions suivantes est vraie*

1. *Pour* $i = 0, 2$, *le coefficient du terme de premier ordre de* a_i *est différent de celui de* b_i,
2. $\alpha \neq \beta$,
3. $\alpha = \beta < 0$,

alors (7.4.5) *admet au plus six solutions positives non dégénérées.*